WHAT'S THE USE?

The Unreasonable Effectiveness of Mathematics

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Unreasonable Effectiveness

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure even though perhaps also to our bafflement, to wide branches of learning.
>
> Eugene Wigner, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*

What is mathematics for?

What is it doing for *us*, in our daily lives?

Not so long ago, there were easy answers to these questions. The typical citizen used basic arithmetic all the time, if only to check the bill when shopping. Carpenters needed to know elementary geometry. Surveyors and navigators needed trigonometry as well. Engineering required expertise in calculus.

Today, things are different. The supermarket checkout totals the bill, sorts out the special meal deal, adds the sales tax. We listen to the beeps as the laser scans the barcodes, and as long as the beeps match the goods, we assume the electronic gizmos know what they're doing. Many professions still rely on extensive mathematical knowledge, but even there, we've outsourced most of the mathematics to electronic devices with built-in algorithms.

My subject is conspicuous by its absence. The elephant isn't even in the room.

It would be easy to conclude that mathematics has become outdated and obsolete, but that view is mistaken. Without mathematics, today's world would fall apart. As evidence, I'm going to show you applications to politics, the law, kidney transplants, supermarket delivery schedules, Internet security, movie special effects, and making springs. We'll see how mathematics plays an essential role in medical scanners, digital photography, fibre broadband, and satellite navigation. How it helps us predict the effects of climate change; how it can protect us against terrorists and Internet hackers.

Remarkably, many of these applications rely on mathematics that originated for totally different reasons, often just the sheer fascination of following your nose. While researching this book I was repeatedly surprised when I came across uses of my subject that I'd never dreamed existed. Often they exploited topics that I wouldn't have expected to have practical applications, like space-filling curves, quaternions, and topology.

Mathematics is a boundless, hugely creative system of ideas and methods. It lies just beneath the surface of the transformative technologies that are making the twenty-first century totally different from any previous era – video games, international air travel, satellite communications, computers, the Internet, mobile phones.[1] Scratch an iPhone, and you'll see the bright glint of mathematics.

Please don't take that literally.

*

There's a tendency to assume that computers, with their almost miraculous abilities, are making mathematicians, indeed mathematics itself, obsolete. But computers no more displace mathematicians than the microscope displaced biologists. Computers change the way we go about *doing* mathematics, but mostly they relieve us of the tedious bits. They give us time to think, they help us search for patterns, and they add a powerful new weapon to help advance the subject more rapidly and more effectively.

In fact, a major reason why mathematics is becoming ever more essential is the ubiquity of cheap, powerful computers. Their rise has opened up new opportunities to apply mathematics to real-world issues. Methods that were hitherto impractical, because they needed too many calculations, have now become routine. The greatest mathematicians of the pencil-and-paper era would have flung up their hands in despair at any method requiring a billion calculations. Today, we routinely use such methods, because we have technology that can do the sums in a split second.

Mathematicians have long been at the forefront of the computer revolution – along with countless other professions, I hasten to add. Think of George Boole, who pioneered the symbolic logic that forms the basis of current computer architecture. Think of Alan Turing, and his universal Turing machine, a mathematical system that can compute anything that's computable. Think of Muhammad al-Khwarizmi, whose algebra text of AD 820 emphasised the role of systematic computational procedures, now named after him: *algorithms*.

Most of the algorithms that give computers their impressive abilities are firmly based on mathematics. Many of the techniques concerned have been taken 'off the shelf' from the existing store of mathematical ideas, such as Google's PageRank algorithm, which quantifies how important a website is and founded a multibillion dollar industry. Even the snazziest deep learning algorithm in artificial intelligence uses tried and tested mathematical concepts such as matrices and weighted graphs. A task as prosaic as searching a document for a particular string of letters involves, in one common method at least, a mathematical gadget called a finite-state automaton.

The involvement of mathematics in these exciting developments tends to get lost. So next time the media propel some miraculous new ability of computers to centre stage, bear in mind that hiding in the wings there will be a lot of mathematics, *and* a lot of engineering, physics, chemistry, and psychology as well, and that without the support of this hidden cast of helpers, the digital superstar would be unable to strut its stuff in the spotlight.

*

The importance of mathematics in today's world is easily underestimated because nearly all of it goes on behind the scenes. Walk along a city street and you're overwhelmed by signs proclaiming the daily importance of banks, greengrocers, supermarkets, fashion outlets, car repairs, lawyers, fast food, antiques, charities, and a thousand other activities and professions. You don't find a brass plaque announcing the presence of a consulting mathematician. Supermarkets don't sell you mathematics in a can.

Dig a little deeper, however, and the importance of mathematics quickly becomes apparent. The mathematical equations of aerodynamics are vital to aircraft design. Navigation depends on trigonometry. The way we use it today is different from how Christopher Columbus used it, because we embody the mathematics in electronic devices instead of pen, ink, and navigation tables, but the underlying principles are much the same. The development of new medicines relies on statistics to make sure the drugs are safe and effective. Satellite communications depend on a deep understanding of orbital dynamics. Weather forecasting requires the solution of equations for how the atmosphere moves, how much moisture it contains, how warm or cold it is, and how all of those features interact. There are thousands of other examples. We don't notice they involve mathematics, because we don't need to know that to benefit from the results.

What makes mathematics so useful, in such a broad variety of human activities?

It's not a new question. In 1959 the physicist Eugene Wigner gave a prestigious lecture at New York University,[2] with the title 'The Unreasonable Effectiveness of Mathematics in the Natural Sciences'. He focused on science, but the same case could have been made for the unreasonable effectiveness of mathematics in agriculture, medicine, politics, sport ... you name it. Wigner himself hoped that this effectiveness would extend to 'wide branches of learning'. It certainly did.

The key word in his title stands out because it's a surprise: *unreasonable*. Most uses of mathematics are entirely reasonable, once you find out which methods are involved in solving an important problem or inventing a useful gadget. It's entirely reasonable, for instance, that engineers use the equations of aerodynamics to help them design aircraft. That's what aerodynamics was created for in the first place. Much of the mathematics used in weather forecasting arose with that purpose in mind. Statistics emerged from the discovery of large-scale patterns in data about human behaviour. The amount of mathematics required to design spectacles with varifocal lenses is huge, but most of it was developed with optics in mind.

The ability of mathematics to solve important problems becomes unreasonable, in Wigner's sense, when no such connection exists between the original motivation for developing the mathematics, and the eventual application. Wigner began his lecture with a story, which I'll paraphrase and embellish slightly.

Two former school classmates met up. One, a statistician working on population trends, showed the other one of his research papers, which began with a standard formula in statistics, the normal distribution or 'bell curve'.[3] He explained various symbols – this one is the population size, that one is a sample average – and how the formula can be used to infer the size of the population without having to count everyone. His classmate suspected his friend was joking, but he wasn't entirely sure, so he asked about other symbols. Eventually he came to one that looked like this: π.

'What's that? It looks familiar.'

'Yes, it's *pi* – the ratio of the circumference of the circle to its diameter.'

'Now I know you're pulling my leg,' said the friend. 'What on earth can a circle have to do with population sizes?'

The first point about this story is that the friend's scepticism was entirely sensible. Common sense tells us that two such disparate concepts can't possibly be related. One is about geometry, the other about people, for heaven's sake. The second point is that despite common sense, there's a connection. The bell curve has a formula, which happens to involve the number π. It's not just a convenient approximation; the exact number really is good old familiar π. But the reason it appears in the context of the bell curve is far from intuitive, even to mathematicians, and you need advanced calculus to see how it arises, let alone *why*.

Let me tell you another story about π. Some years ago we had the downstairs bathroom renovated. Spencer, an amazingly versatile craftsman who came to fit the tiles, discovered that I wrote popular mathematics books. 'I've got a maths problem for you,' he said. 'I've got to tile a circular floor, and I need to know its area to work out how many tiles I'll need. There was some formula they taught us …'

'Pi *r* squared,' I replied.

'That's the one!' So I reminded him how to use it. He went away happy, with the answer to his tiling problem, a signed copy of one of my books, and the discovery that the mathematics he'd done at school was, contrary to his long-held belief, useful in his present occupation.

The difference between the two stories is clear. In the second story, π turns up because it was introduced to solve exactly that kind of problem in the first place. It's a simple, direct story about the effectiveness of mathematics. In the first story, π also turns up and solves the problem, but its presence is a surprise. It's a story of *unreasonable* effectiveness: an application of a mathematical idea to an area totally divorced from that idea's origins.

*

In *What's the Use?* I'm not going to say much about reasonable uses of my subject. They're worthy, they're interesting, they're as much a part of the mathematical landscape as everything else, they're equally important – but they don't make us sit up and say 'Wow!' They can also mislead the Powers That Be into imagining that the only way to advance mathematics is to decide on the problems and then get the mathematicians to invent ways to solve them. There's nothing wrong with goal-oriented research of this kind, but it's fighting with one arm tied behind your back. History repeatedly shows the value of the second arm, the amazing scope of human imagination. What gives mathematics its power is the *combination* of these two ways of thinking. Each complements the other.

For instance, in 1736, the great mathematician Leonhard Euler turned his mind to a curious little puzzle about people taking walks across bridges. He knew it was interesting, because it seemed to require a new kind of geometry, one that abandoned the usual ideas of lengths and angles. But he couldn't possibly have anticipated that in the twenty-first century the subject that his solution kick-started would help more patients get life-saving kidney transplants. For a start, kidney transplants would have seemed pure fantasy at that time, but even if they hadn't, any connection with the puzzle would have seemed ridiculous.

And who would ever have imagined that the discovery of space-filling curves – curves that pass through *every* point of a solid square – could help Meals on Wheels to plan its delivery routes? Certainly not the mathematicians who studied such questions in the 1890s, who were interested in how to define esoteric concepts like 'continuity' and 'dimension', and initially found themselves explaining why cherished mathematical beliefs can be wrong. Many of their colleagues denounced the entire enterprise as wrong-headed and negative. Eventually everyone realised that it's no good living in a fool's paradise, assuming that everything will work perfectly when in fact it won't.

It's not just the mathematics of the past that gets used in this way. The kidney transplant methods rely on numerous modern extensions of Euler's original insight, among them powerful algorithms for combinatorial optimisation – making the best choice from a huge range of possibilities. The myriad mathematical techniques employed by movie animators include many that go back a decade or less. An example is 'shape space', an infinite-dimensional space of curves that are considered to be the same if they differ only by a change of coordinates. They're used to make animation sequences appear smoother and more natural. Persistent homology, another very recent development, arose because pure mathematicians wanted to compute complicated topological invariants that count multidimensional holes in geometric shapes. Their method also turned out to be an effective way to ensure that sensor networks provide full coverage when protecting buildings or military bases against terrorists or other criminals. Abstract concepts from algebraic geometry – 'supersingular isogeny graphs' – can make Internet communications secure against quantum computers. These are so new that they currently exist only in rudimentary form, but they will trash today's cryptosystems if they can fulfil their potential.

Mathematics doesn't just spring such surprises on rare occasions. It makes a positive habit of it. In fact, as far as many mathematicians are concerned, these surprises are the most interesting uses of the subject, and the main justification for considering it to *be* a subject, rather than just a rag-bag of assorted tricks, one for each kind of problem.

Wigner went on to say that 'the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and … there is no rational explanation for it.' It is, of course, true that mathematics grew out of problems in science in the first place, but Wigner wasn't puzzled by the subject's effectiveness in areas it was designed for. What baffled him was its effectiveness in apparently unrelated ones. Calculus grew from Isaac Newton's research on the motion of the planets, so it's not greatly surprising that it helps us to understand how planets move. However, it *is* surprising when calculus lets us make statistical estimates of human populations, as in Wigner's little story, explains changes in the numbers of fish caught in the Adriatic Sea during the First World War,[4] governs option pricing in the financial sector, helps engineers to design passenger jets, and is vital for telecommunications. Because calculus wasn't invented for any such purpose.

Wigner was right. The way mathematics repeatedly turns up uninvited in the physical sciences, and in most other areas of human activity, is a mystery. One proposed solution is that the universe is 'made of' mathematics, and humans are just digging out this basic ingredient. I'm not going to argue the toss here, but if this explanation is correct it replaces one mystery by an even deeper one. *Why* is the universe made of mathematics?

*

On a more pragmatic level, it can be argued that mathematics has several features that help to make it unreasonably effective in Wigner's sense. One is, I agree, its many links with natural science, which transfer to the human world as transformative technology. Many of the great mathematical innovations have indeed arisen from scientific enquiries. Others are rooted in human concerns. Numbers arose from basic accountancy (how many sheep have I got?). Geometry *means* 'earth-measurement', and was closely related to the taxation of land and in ancient Egypt to the construction of pyramids. Trigonometry emerged from astronomy, navigation, and map-making.

However, that alone isn't an adequate explanation. Many other great mathematical innovations have *not* arisen from scientific enquiry or specific human issues. Prime numbers, complex numbers, abstract algebra, topology – the primary motivation for these discoveries/inventions was human curiosity and a sense of pattern. This is a second reason why mathematics is so effective: mathematicians use it to seek patterns and to tease out underlying structure. They search for *beauty*, not of form but of logic. When Newton wanted to understand the motion of the planets, the solution came when he thought like a mathematician and looked for deeper patterns beneath the raw astronomical data. Then he came up with his law of gravity.[5] Many of the greatest mathematical ideas had no real-world motivation at all. Pierre de Fermat, a lawyer who did mathematics for fun in the seventeenth century, made fundamental discoveries in number theory: deep patterns in the behaviour of ordinary whole numbers. It took three centuries for his work in that area to acquire practical applications, but right now the commercial transactions that drive the Internet wouldn't be possible without it.

Another feature of mathematics that's become increasingly evident since the late 1800s is *generality*. Different mathematical structures have many common features. The rules of elementary algebra are the same as those of arithmetic. Different kinds of geometry (Euclidean, projective, non-Euclidean, even topology) are all closely related to each other. This hidden unity can be made explicit by working, from the start, with general structures that obey specified rules. Understand the generalities, and all the special examples become obvious. This saves a lot of effort, which would otherwise be wasted doing essentially the same thing many times in slightly different language. It has one downside, however: it tends to make the subject more abstract. Instead of talking about familiar things such as numbers, the generalities must refer to anything obeying the same *rules* as numbers, with names like 'Noetherian ring', 'tensor category', or 'topological vector space'. When this kind of abstraction is carried to extremes, it can become difficult to understand what the generalities *are*, let alone how to make use of them. Yet they're so useful that the human world would no longer function without them. You want Netflix? Someone has to do the maths. It's not magic; it just feels like it.

A fourth feature of mathematics, highly relevant to this discussion, is its *portability*. This is a consequence of its generality, and it's why abstraction is necessary. Irrespective of the problem that motivated it, a mathematical concept or method possesses a level of generality that often makes it applicable to quite different problems. Any problem that can be recast in the appropriate framework then becomes fair game. The simplest and most effective way to create portable mathematics is to design portability in from the start, by making the generalities explicit.

For the last two thousand years, mathematics has taken its inspiration from three main sources: the workings of nature, the workings of humanity, and the internal pattern-seeking tendencies of the human mind. These three pillars support the entire subject. The miracle is that despite its multifarious motivations, mathematics is *all one thing*. Every branch of the subject, whatever its origins and aims, has become tightly bound to every other branch – and the bonds are becoming ever stronger and ever more entangled.

This points to a fifth reason why mathematics is so effective, and in such unexpected ways: its *unity*. And alongside this goes a sixth, for which I'll provide ample evidence as we proceed: its *diversity*.

Reality, beauty, generality, portability, unity, diversity. Which, together, lead to utility.

It's as simple as that.

2

How Politicians Pick Their Voters

> Ankh-Morpork had dallied with many forms of government and had ended up with that form of democracy known as One Man, One Vote. The Patrician was the Man; he had the Vote.
>
> Terry Pratchett, *Mort*

The ancient Greeks gave the world many things – poetry, drama, sculpture, philosophy, logic. They also gave us geometry and democracy, which have turned out to be more closely linked than anyone might have expected, least of all the Greeks. To be sure, the political system of ancient Athens was a very limited form of democracy; only free men could vote, not women or slaves. Even so, in an era dominated by hereditary rulers, dictators, and tyrants, Athenian democracy was a distinct advance. As was Greek geometry, which, in the hands of Euclid of Alexandria, emphasised the importance of making your basic assumptions clear and precise, and deriving everything else from them in a logical and systematic fashion.

How on earth can mathematics be applied to politics? Politics is about human relationships, agreements, and obligations, whereas mathematics is about cold, abstract logic. In political circles, rhetoric trumps logic, and the inhuman calculations of mathematics seem far removed from political bickering. But democratic politics is carried out according to rules, and rules have consequences that aren't always foreseen when they're first drawn up. Euclid's pioneering work in geometry, collected in his famous *Elements*, set a standard for deducing consequences from rules. In fact, that's not a bad definition of mathematics as a whole. At any rate, after a mere 2,500 years, mathematics is beginning to infiltrate the political world.

One of the curious features of democracy is that politicians who claim to be devoted to the idea that decisions should be made by 'The People' repeatedly go out of their way to ensure that this doesn't happen. This tendency goes right back to the first democracy in ancient Greece, where the right to vote was given only to adult male Athenians, about one third of the adult population. From the moment the idea of electing leaders and selecting policy by popular vote was conceived, so was the even more attractive idea of subverting the entire process, by controlling who voted and how effective their votes are. This is easy, even when every voter gets one vote, because the effectiveness of a vote depends on the context in which it's cast, and you can rig the context. As journalism professor Wayne Dawkins delicately put it, this amounts to politicians picking their voters instead of voters picking their politicians.[6]

That's where mathematics comes in. Not in the cut-and-thrust of political debate, but in the structure of the debating rules and the context in which they apply. Mathematical analysis cuts both ways. It can reveal new, cunning methods for rigging votes. It can also shine a spotlight on such practices, providing clear evidence of that kind of subversion, which can sometimes be used to prevent it happening.

Mathematics also tells us that any democratic system must involve elements of compromise. You can't have everything you want, however desirable that might be, because the list of desirable attributes is self-contradictory.

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On 26 March 1812 the *Boston Gazett*e gave the world a new word: *gerrymander*. Originally spelt Gerry-mander, it's what Lewis Carroll later called a portmanteau word, created by combining two standard words. 'Mander' was the tail end of 'salamander', and 'Gerry' was the tail end of Elbridge Gerry, governor of Massachusetts. We don't know for sure who first put the two tails together, but on circumstantial grounds, historians tend to opt for one of the newspaper's editors, Nathan Hale, Benjamin Russell, or John Russell. Incidentally, 'Gerry' was pronounced with a hard G, like 'Gary', but 'gerrymander' has a soft G, like 'Jerry'.

What was Elbridge Gerry doing that got him combined with a lizard-like creature, which, in medieval folklore, was reputed to dwell in fire?

Rigging an election.

More precisely, Gerry was responsible for a bill that redrew the district boundaries in Massachusetts for elections to the state senate. Districting, as it's called, naturally leads to boundaries; it is, and has long been, common in most democracies. The overt reason is practicality: it's awkward to take decisions if the entire nation gets to vote on every proposal. (Switzerland comes close: up to four times a year the Federal Council chooses proposals for citizens to vote on, essentially a series of referendums. On the other hand, women didn't get the vote there until 1971, and one canton held out until 1991.) The time-honoured solution is for voters to elect a much smaller number of representatives, and let the representatives make the decisions. One of the fairer methods is proportional representation: the number of representatives of a given political party is proportional to the number of votes that that party receives. More commonly, the population is divided into districts, and each district elects a number of representatives, roughly proportional to the number of electors in that district.

For example, in American presidential elections, each state votes for a specific number of 'electors' – members of the Electoral College. Each elector has one vote, and who becomes President is decided by a simple majority of these votes. It's a system that originated when the only way to get a message from the American hinterland to the centres of power was to carry a letter on horseback or in a horse-drawn coach. Long-distance rail and the telegraph came later. In those days, totalling up the votes of huge numbers of individuals was too slow.[7] But this system also ceded control to the elite members of the Electoral College. In British parliamentary elections, the country is divided into (mainly geographical) constituencies, each of which elects one Member of Parliament (MP). Then the party (or combination of parties in a coalition) with the most MPs forms the government, and chooses one of its MPs to be Prime Minister, by a variety of methods. The Prime Minister has considerable powers, and in many ways is more like a President.

There's also a covert reason for funnelling democratic decisions through a small number of gatekeepers: it's easier to rig the vote. All such systems have innate flaws, which often lead to strange results, and on occasion they can be exploited to flout the Will of the People. In several recent US presidential elections, the total number of votes cast by the People for the candidate who lost was greater than the number of votes for the candidate who won. Agreed, the current method for choosing a President doesn't depend on the popular vote, but with modern communications the only reason not to change to a fairer system is that a lot of powerful people prefer it the way it is.

The underlying problem here is 'wasted votes'. In each state, a candidate needs half the total plus one vote (or half a vote if the total is odd) to win; any extra votes beyond that threshold make no difference to what happens at the Electoral College stage. Thus, in the 2016 presidential election, Donald Trump received 304 votes in the Electoral College compared to 227 for Hillary Clinton, but Clinton's popular vote exceeded Trump's by 2·87 million. Trump thereby became the fifth US President to be elected while losing the popular vote.

The Gerry-mander, thought to have been drawn in 1812 by Elkanah Tisdale.

The boundaries of American states are effectively immutable, so this is not a districting issue. In other elections, the boundaries of districts can be redrawn, usually by the party in power, and a more insidious flaw appears. Namely, that party can draw the boundaries to ensure that unusually large numbers of votes for the opposing party are wasted. Cue Elbridge Gerry and the senate vote. When Massachusetts voters saw the map of electoral districts, most of them looked entirely normal. One didn't. It combined twelve counties from the west and north of the state into a singe, sprawling, meandering region. To the political cartoonist responsible for the drawing that shortly appeared in the *Boston Gazette* – probably the painter, designer, and engraver Elkanah Tisdale – this district closely resembled a salamander.

Gerry belonged to the Democratic-Republican Party, which was in competition with the Federalists. In the 1812 election the Federalists won the House and governorship in the state, which put Gerry out of office. However, his redistricting of the state senate worked a treat, and it was held comfortably by the Democratic-Republicans.

*

The mathematics of gerrymandering begins by looking at how people do it. There are two main tactics, packing and cracking. *Packing* spreads your own vote as evenly as possible, with a small but decisive majority, in as many districts as possible, and cedes the rest to the enemy. Sorry, opposition. *Cracking* breaks up the opposition's votes so that they lose as many districts as possible. *Proportional representation*, in which the number of representatives is proportional to each party's total votes (or as close to that as possible given the numbers) avoids these tricks, and is fairer. Unsurprisingly, the US constitution makes proportional representation illegal, because as the law stands, districts must have only one representative. In 2011 the UK held a referendum on another alternative, the single transferable vote: the people voted against this change. There's never been a referendum on proportional representation in the UK.

Here's how packing and cracking work, in an artificial example with very simple geography and voting distributions.

The state of Jerimandia is contested by two political parties, the Lights and the Darks. There are fifty regions, to be split into five districts. In recent elections, Light has a majority in twenty