MATHEMATICAL INTELLIGENCE

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What we have that machines don't

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For Leena,
my pride *and* my joy.

Contents

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Introduction

THE CASE FOR MATHEMATICAL INTELLIGENCE

MIT, 1950s. The first wave of Artificial Intelligence is on the horizon. Marvin Minsky, one of the field's leading figures, proclaims: 'We're going to make machines intelligent. We're going to make them conscious.' Douglas Engelbart, a peer of Minsky's, retorts: 'You're going to do all that for the machines? What are you going to do for the people?'[1]

Artificial intelligence (AI) researchers are nothing if not bullish about the prospects of their creations. The field kicked off in earnest in 1956 at a summer workshop held at Dartmouth College, New Hampshire, where the founding fathers of AI set out their vision in no uncertain terms. Intelligent machines, they believed, were to propel humanity into the next golden age of innovation by 'simulating every aspect of learning or any other feature of intelligence'.[2] The timeframe was bolder still: one summer was all they would need to break the back of AI.

Things turned out to be rather more complicated, as a summer of hype gave way to a succession of AI winters, with progress in the field largely stagnant for several decades. But if you've caught the headlines recently, you'll know that AI is currently the subject of renewed hype. Between flagship triumphs in popular games, the growing presence of home assistants, and the coming of self-driving cars, the machines have resumed their rise.

We humans have distinguished ourselves from other species by inventing tools to help us solve our most challenging problems. And yet we may be complicit in our own demise because some of these tools have become so powerful that they appear to pose genuine threats to our ways of thinking and being. Studies of the growing threat of automation to human labour abound,[3] while the so-called 'superintelligent' machines of tomorrow may force us to re-examine what it even means to be human in the first place.

As we enter this new cycle of ratcheted expectations, hopes and anxieties around the latest wave of technological innovation, Engelbart's question should resonate loud and clear. We reserve such reverence for technology that we risk overlooking our own human capabilities. Machines lack some of the basic qualities of human thinking – qualities we have sidelined through our mechanistic ways of schooling and working, and qualities that we need to urgently reawaken to thrive alongside our silicon counterparts.

As it happens, humans have – through millions of years of evolution and thousands of years of continual refinement – developed a powerful system for making sense of the world, for imagining new ones, and for devising and solving complex problems. This system has helped us create the economies that underpin our society. It has shaped our notions of democracy. It has spawned technologies that now stare us down, but the same system can equip us with the skills to tame these digital beasts.

The system has a name: *mathematics*.

What is mathematics, really?

Mathematics has been described as an art, a language and a science. For some, it is a means of unlocking nature's secrets. As Galileo testified so eloquently: '[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language.' This is mathematics as the language of the universe, the engine of scientific progress.

The scope of mathematics transcends our physical universe. Entire swathes of the subject are explored for their own sake, driven by the deep satisfaction that comes from dreaming up new concepts, piecing together ideas, and grappling with thorny problems. Many mathematicians seek out aesthetic qualities in their craft. The twentieth-century mathematician and philosopher Bertrand Russell spoke of the subject's 'supreme beauty – a beauty cold and austere … capable of a stern perfection such as only the greatest art can show'.[4] Many see themselves as artists as well as scientists – 'makers of patterns',[5] to borrow a description from G. H. Hardy, a contemporary of Russell's. It is not uncommon for mathematicians to deride the need to apply their thinking to the 'real' world, as if utility were some kind of distraction. It has even been proposed that some aspects of mathematical inquiry have a hedonistic basis.[6]

From these varied motives, mathematics is often partitioned into two supposed types: there is *applied* mathematics, which, as the name suggests, is concerned with problems of the real world. Then there is the presumptively labelled strand of *pure* mathematics, which centres on more abstract concepts and rigorous arguments often removed from practical consideration. This separation is felt keenly at university, where maths students are expected to declare their allegiances before specialising in one area. I was of the *pure* persuasion. Yet, since leaving formal mathematics a decade ago, much of my work has been rooted in datasets and algorithms – about as *applied* as it comes.

Having bridged the pure/applied divide, I have come to realise that it is an arbitrary and limiting way of characterising the subject. There is a commonality that binds mathematicians of all types. Without exception, we derive immense joy from tackling maths problems, a satisfaction akin to solving our favourite puzzles. Mathematics is even alleged to elicit the same physiological reactions as sexual activity (yes, really).[7] Alongside that pleasure comes power; whatever branch of mathematics a mathematician happens to be probing, they are using the mind's highest faculties and building a store of portable mental models that serve them in all parts of life.

It may feel risky to invest time and effort in studying mathematics based on nebulous notions of pleasure and power. But mathematics cannot help but bring practical uses too. It is not at all unusual for a field of mathematics that starts out as pure intellectual inquiry to later find itself in practical settings. Prime numbers (whole numbers greater than 1 that cannot be divided into smaller whole number parts) were first studied for their unusual arithmetic properties, yet internet security now relies on them – your credit card details are kept secure by the sheer difficulty of finding the prime factors of really large numbers. The Greeks were enthralled by the geometric properties of ellipses; only centuries later would Kepler discover that planets move around the sun in an elliptical orbit. The topology of knots, a delight to study in its own right, has applications in protein folding. And calculus (the study of continuous change), arguably the most applied of all mathematical topics, which was the basis for Newton's study of planetary motion, and whose tools are indispensable to engineers, physicists, financial analysts, even historians,[8] was developed within the rigorous frameworks of pure mathematics. I could go on.

The theoretical physicist Eugene Wigner encapsulated this entwining of intellectual curiosity and utility by remarking on what he called the 'unreasonable effectiveness' of mathematics, declaring that 'the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and there is no rational explanation for it'.[9]

The 'usefulness' of mathematics is not limited to specific real-world applications. It arises chiefly from its invitation to explore a vast range of concepts, even arcane ones. Mathematics transports us into multiple worlds, each governed by its own rules. It encourages us to break free of convention and leap from one conceptual system to another. These faraway worlds can also train us to think in ways that enrich our understanding of our own, physical one. Even as the content of my own doctorate in pure mathematics drifts from memory (to the point where I can scarcely grasp its essential ideas any longer),[10] the process by which it was created remains its most enduring contribution to my everyday thinking and problem solving.

Mathematical intelligence is not about calculus or topology any more than musical intelligence is limited to a particular genre or instrument. It is a system for making us better thinkers and problem solvers, using the proven tools of mathematicians. And in the age of smart machines, it is needed more than ever.

Mathematics and calculation: a false coupling

The mathematics I've just described is quite apart from what we encounter at school. 'School mathematics' places great emphasis on calculation. A calculation is a routine operation performed on certain objects, often numbers, to produce a particular result. It can be as simple as counting and as complicated as Google's search-ranking algorithms (an algorithm here just means a list of step-by-step instructions).* School mathematics is premised on the idea that rehearsing a litany of routine calculational techniques is a strict prerequisite for mathematical intelligence, and a gateway to employment. Topics such as calculus, algebra and geometry, each of which contain a multitude of rich concepts, are stripped down to bare calculational form.

The marriage between mathematics and calculation is the result of several forces. The first is an industrial paradigm of formal education whose roots can be traced back to the mid-nineteenth century, when the aims of mass schooling coalesced with notions of mechanisation and scale, and increases in urban populations fuelled demand for everyday numeracy skills such as counting money and telling the time. As universal education systems sprang up the world over, subject matter reflected the needs of a mathematically literate workforce. In England, for instance, arithmetic dominated the curriculum, and additional topics – such as algebra, mechanics and fractions – were introduced with the goals of employment in mind.[11]

Society has made giant leaps of progress since then, yet school mathematics has remained largely static; national and international curriculum standards remain heavily couched in speed and proficiency with calculation. The stubborn persistence of calculation in education also owes a debt to widely held beliefs around the nature of mathematics. *Platonism*

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*Computation and calculation have slightly different meanings. The former tends to refer to algorithmic processes, the latter to arithmetic ones. I will use them interchangeably because they both espouse the same kinds of routine thinking processes.

– first espoused by the Greek philosopher Plato – holds that mathematical objects are abstract entities independent of language, thought or practices. Just as electrons and planets exist independently of us, so do mathematical concepts such as number. In this view, there is a single form of mathematics, timeless and immutable. Alongside Platonism, there is the *formalist* view, which gained traction in the twentieth century and considers mathematics a self-contained system of logical truths, each derivable from first principles. The Platonist and formalist philosophies, especially popular among 'pure' mathematicians, conspire to reduce mathematics to a single pathway of predetermined, hard-coded truths. Abstraction is the gold standard in this framing of mathematics, its raison d'être, best accessed by mastering symbol manipulation. The execution of mathematical procedures – fast, precise calculation – is seen as the single pathway to deep mathematical thought.

The Platonist–formalist view overlooks the crucial fact that mathematics takes on rich and diverse forms,[12] all of which are birthed in the context of local environment and experience.[13] Take something as seemingly universal as our number system. It arises out of a series of choices, from the symbols we use to denote quantities to how we group together objects to manage large amounts, to how we perform arithmetic on numbers. In schools across the world, students are taught Hindu–Arabic numerals (0, 1, 2, and so on), the decimal system (grouping numbers into tens), and specific algorithms for performing addition, subtraction, multiplication and division. Students are led to believe that these choices are inevitable – the only conceivable way to think about numbers – when in fact they are situated within a historical and sociocultural backdrop. As we'll see in later chapters, communities around the world to this day adopt highly varied representations for numbers.

Mathematics in the real world is more situational and contextual than Platonism and formalism would suggest.

My work has taken me to classrooms the world over and I can confirm that, despite its short-sightedness, the Platonist–formalist ideal is alive and well everywhere. A common thread binds the mathematics taught to marginalised communities in Kenya, children of Microsoft executives in Washington State, students of Eton College, and low-income families in rural Mexico. In all these cases, school mathematics is characterised by a heavy diet of calculation,[14] and mathematical talent is conceived as the ability to execute these techniques flawlessly and at speed.

School mathematics comes wrapped in the promise that this very particular skill set will, on some unspecified date in the future, serve students' everyday needs. That promise may have held up in the nineteenth century, when, for example, the formulae of trigonometry would guide your career as a carpenter or surveyor or navigator, and you would be expected to make the requisite calculations by hand. Yet the twenty-first-century student will discover, if they haven't already, that calculation is no longer the unique marker of human mathematical talent. It is almost tautological to say it, but for computation we have computers.

School mathematics is clearly in need of a rethink, which should come as welcome relief to most. Far from evoking the sentiments of wonder or beauty experienced by mathematicians, it is more commonly associated with feelings of dread. In the UK alone, a fifth of the population is afflicted with maths anxiety.[15] For these people, the anticipation and experience of doing mathematics activate the same regions of the brain that give rise to pain.[16] Attitudes towards mathematics have been shown to deteriorate with age,[17] and many people, scarred by their encounters in school, flee into the safe sanctuary of adulthood, resolving never again to confront anything that resembles mathematics. Is the Platonist–formalist method of education simply the price we have to pay to feel the power of mathematics – to appreciate its unreasonable effectiveness? Even if a casualty rate of one in five is deemed palatable, the apparent victors of this brand of mathematics find themselves trapped in a false sense of security. As an admissions tutor at Oxford University, and more recently as an employer, I have interviewed hundreds of candidates who naïvely presume that a clean sweep of top grades in mathematics at school has prepared them to think creatively and tackle complex problems.

The German poet Hans Magnus Enzensberger has described mathematics as 'a blind spot in our culture – alien territory, in which only the elite, the initiated few have managed to entrench themselves'.[18] There is a yawning chasm between the mathematics enjoyed by professional mathematicians and the monotony of most school curricula.

Professional mathematicians, for their part, tend to keep calculation at arm's length. They recognise that techniques such as long division, the quadratic formula and trigonometric identities occupy a small space within the mathematical landscape, a tiny sliver of all the concepts available in the subject. Entire branches of mathematics are removed from calculation, and even where calculations surface, the creative elements of mathematical intelligence reside in dreaming up such methods in the first place, understanding their inner workings and applying them in novel settings. The specific act of calculation is secondary and offers little joy or illumination.

New calculating tools, new mathematics

The history of mathematics shares a timeline with an ongoing effort to liberate humans from the tedium of calculation. Performing calculations does not come naturally to us. Time and again, we have created tools and technologies that outsource the most mechanical aspects of mathematics.

Great leaps have been made with leading-edge calculating tools.[19] Where our earliest ancestors marshalled pebbles and grains to keep track of basic quantities, the city planners of Babylonia, Sumeria and Egypt used formal calculation schemes which were brought to bear on problems of engineering, land administration, astronomy, timekeeping, planning and logistics. Calculation, along with reading and writing, became a cornerstone of more developed civilisations. Some of the earliest surviving government records are replete with calculations central to administration.

Physical counting instruments were always close at hand. The abacus that helps us count large quantities has its roots in the pebble-counting schemes of the ancient Romans, and as calculations grew in complexity, so too did the power of our tools. Older readers may recall using a slide rule in school to assist in weighty calculations such as the multiplication of large numbers. The slide rule was based on John Napier's logarithm tables. Napier was born into a Scottish family of estate owners in 1550. Copernicus had just developed the heliocentric model of the universe, placing the sun at its centre for the first time. Columbus had sailed the Atlantic, and Renaissance artists were advancing their own frontiers. Yet the world remained heavily dependent on tired calculational conventions. The work of masons, merchants, navigators and astronomers all required methods of long division and multiplication that were tediously handcrafted, prone to human error, and prohibitively expensive to carry out (pen and paper did not come cheap).

On his travels across Europe as a young student, Napier observed the burden of calculation first-hand. He would encounter decorative books composed solely of mathematical tables and currency versions, created and used daily by merchants. The tables still demanded a hefty degree of calculation on the part of the user. There had to be a more effective way, Napier thought, of removing what he called 'those hindrances' to trade. Napier was alluding to what cognitive psychologists now term our 'working memory', which handles short-term information and is limited to between four and seven objects at a time.[20] This makes multistep calculations such as long multiplication or division difficult to perform, as we strain to keep track of each moving part.

In his famous work *Mirifici Logarithmorum Canonis Descriptio* ('Description of the marvellous canon of logarithms'), Napier introduced a powerful mathematical object called the logarithm function. To grasp the intuition behind logarithms, first consider a familiar multiplication involving powers of 10:

$$\underbrace{100}_{\substack{2\\\text{zeros}}} \times \underbrace{1000}_{\substack{3\\\text{zeros}}} = \underbrace{100000}_{\substack{5\\\text{zeros}}}$$

$$2 + 3 = 5$$

This calculation is straightforward because we just 'add the zeros' in each term to get our product. It would be handy if every multiplication could be managed in such a simple way. Napier's logarithm makes this possible. In the numbers above, the string of zeros corresponds to how many times 10 multiplies by itself – twice for 100, thrice for 1000, and so on. With this in mind, the logarithm of a number is defined by how many times you have to multiply 10 by itself to get that number. So the logarithm of 100, denoted log(100), is 2, and the logarithm of 1000, denoted log(1000), is 3.

The clever, mathematical part is that the logarithm can be defined for every positive number, not just powers of 10. The logarithm of 95 is 1.978, the logarithm of 2367 is 3.374, and the logarithm of 3 is 0.477, which is to say that 'if you multiply 10 by itself 0.477 times you will get 3'. That may sound strange at first, but the conceptual power of mathematical functions allows us to bring such notions into existence.

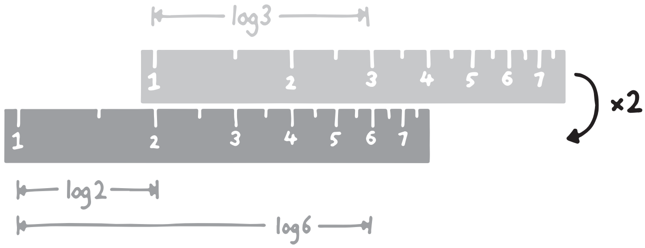
A useful property of logarithms is that they obey the following rule:

$$\log(a \times b) = \log(a) + \log(b)$$

Suppose we want to multiply two large numbers. Napier explained how, using the above formula, we can transform the problem into one involving addition, which is simpler and less error-prone. All we need is a table that lists the 'logarithmic value' of each number. The process then goes as follows:

1. Look up the logarithm of each value to be multiplied.
2. Add these two logarithmic values to get a total.
3. Look up the value of the number whose logarithm corresponds to this total. The number you have found is the product of your two original numbers.*

*This method is a slight simplification of how Napier's tables were constructed, but close enough to give reasonable approximations, which is often all we need. It uses the *base 10* logarithm, which can be substituted for any other value – the natural logarithm that is now popular calls on base *e*, where *e* denotes Euler's number.



A slide rule in action: if we slide the top ruler 2 units along the ruler below (a length of log 2) then every number on the ruler below corresponds to multiplying the number above it by 2. For example, the number 3 (which is a length of log 3 along the top ruler) lines up with the number 6 (which is a length of log 6 along the bottom ruler), telling us that 3 × 2 = 6.

Napier's *Canon* comprised huge lists of numbers and their corresponding logarithmic values. It took some twenty years to compile. When dedicating the work to the future King Charles I, Napier wrote of how 'this new course … doth clean take away the difficulty that heretofore hath been in mathematical calculations, and is so fitted to help the weakness of memory'.[21] The slide rule – a compact manifestation of Napier's logarithm tables – appeared in 1654, after his passing. Logarithms can also be exploited to simplify a raft of operations beyond multiplication: powers, square roots and even trigonometric calculations can be closely approximated using simple extensions of the techniques described here, and all of these methods were added to various iterations of the slide rule until the electronic calculator took its place in the latter part of the twentieth century.

Napier's innovation epitomises attempts to automate human effort. For a time, this led to an explosion of jobs. When the eighteenth-century French mathematician and engineer Gaspard de Prony embarked on the project of producing large logarithmic tables for the French Cadastre (the official system of land registration), for which 200,000 logarithms were each to be calculated to upwards of fourteen decimal places, he enlisted a small army of 'human computers' to accomplish the task.[22] De Prony took inspiration from economist Adam Smith's *The Wealth of Nations* and sought to bring Smith's concept of the 'division of labour' to calculation. He imagined a three-tiered pyramid of human labourers. At the top was a small sliver of mathematicians of distinction who devised clever step-by-step instructions – algorithms – for calculating logarithmic values. The second layer consisted of 'algebraists' who would translate these instructions into forms that could easily be computed. The final, most crowded layer consisted of workers who were competent in basic arithmetic and required 'the least knowledge and by far the greatest exertions', performing millions of calculations (addition and subtraction for the most part) and noting the results. In de Prony's model, just two or three mathematicians were needed for every seven or eight algebraists and seventy to eighty workers. With de Prony's labour pyramid, 'big calculation' was born, fashioned in the image of scalable manufacturing.

'Big calculation' trod the same path as manufacturing when it came to mechanisation, as physical calculating machines increasingly took the place of humans. It was against this backdrop that inventor-mathematician Charles Babbage designed two mechanical calculators in the mid-nineteenth century, neither of which was actually constructed during his lifetime (due mainly to financial constraints), but both of which carry huge significance as direct cogwheel-based forerunners of the modern computer. With the emergence of the digital computer and the electronic calculator, Babbage's visions were realised and the era of the *human* computer drew to a close. The heroic swansong of human computers was the 1960s NASA space mission, where the flesh-and-blood calculations of Katherine Johnson and her team helped propel humankind into space.[23]

The work of human computers was profitable in its time, noble even. But calculation has always been the understudy of mathematics (an insight not lost on Johnson and her colleagues, who fought for status in the face of racial and gender prejudice by demonstrating their aptitude for modelling and other essential mathematical skills). Calculation no longer paves a path to employment; today those lower rungs of the pyramid are occupied by machines.

Once computers crept past the calculation feats of humans, they surged ahead and never looked back. The slide rule reigned for over three hundred years, but the electronic calculator that took its place lasted no more than thirty.[24] The competition for pocket-sized electronic calculators was fiercely fought for all of two decades before the advent of the internet and cloud-based technologies. The rapid ascent of computing power was foreseen by Intel co-founder Gordon Moore. In the 1960s, Moore observed that the number of transistors that can be accommodated on a microprocessor seemed to double every eighteen months – an exponential rate. Moore's Law has come to fruition with astonishing accuracy.* By now, our smartphones possess more processing power than the computers and slide rules that sent us to the moon. A world without digital computers is a world without the internet and all that it enables: social media, email, GPS, online shopping, music streaming, remote work, certain kinds of medical diagnosis.

As our calculating tools evolve, so does the nature of

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*One popular interpretation of the trend is that it is a self-fulfilling prophecy: project the growth ahead of time and the software engineering community will rise to the challenge of meeting it.

mathematical work. Writing in the early twentieth century, the British philosopher Alfred Whitehead noted: 'Civilization advances by extending the number of important operations which we can perform without thinking about them.'[25] Just as innovations such as Napier's logarithm tables accelerated scientific discovery in the past, today's technologies are giving rise to whole new ways of doing mathematics.

Over the past few decades, algorithms have evolved significantly in the direction of versatility as well as processing power. A flurry of packages such as Mathematica and Wolfram Alpha have been developed to execute a vast array of procedures. They have birthed new branches of research, such as 'experimental mathematics', where the idea is to study mathematical objects (numbers, shapes and multidimensional vector spaces, to name a few), and the patterns that govern them, through computation. Powerful, automated calculators allow us to make informed guesses and check them through trial and error by crunching through a range of numerical scenarios.

In our everyday lives, too, calculation is as prominent as ever – we analyse offers in the supermarket, mortgage options, calorie counts, and much besides. Getting the best deal (or diet), however, doesn't rest on our number-crunching skills as much as our ability to evaluate information and make sense of data.

With the right tools at our disposal, mathematics gives us all licence to transcend calculation and to think in the most creative ways. As mathematician Keith Devlin put it: 'Calculation was the price we used to have to pay to do mathematics.'[26] Mathematicians have figured out how to use technology to aid their thinking. They've cracked the human–machine conundrum that the rest of society is still grappling with.