THINK LIKE A MATHEMATICIAN

Also by Junaid Mubeen

Mathematical Intelligence:
What We Have That Machines Don't

THINK LIKE A MATHEMATICIAN

Simple Tools for Complex Everyday Problems

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For Elias,
the coolness of my eyes

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Introduction

Here's to maths, and its surprising usefulness

In the summer of 2011, during my final weeks as a maths student at Oxford, I chanced upon an unusual conversation. The features of a maths common room are unmistakable: equations scribbled onto whiteboard tables, animated discussion of highly technical concepts, copious consumption of coffee, a hotly contested game of chess.

As I worked my way through the communal biscuits, my ears pricked up as two of my fellow students shared their latest life updates. What initially caught my attention was the contrast between their two speaking styles: the measured Brit on the one hand, an American who exuded scarcely believable levels of enthusiasm with every syllable on the other. But what held my attention was their topic of conversation.

Brit: I'm spending this weekend with my girlfriend.

American: Wow, sounds serious. Do you think she's the one?

Brit: Well … I believe in $n + 1$.

American: $n + 1$ – that's deep, man. But what does it mean?

Brit: Well, I love her today, and I believe that if I love her at any point in time, then I'll love her at the next point in time. By the principle of mathematical induction, the logical conclusion is that my love for her will endure forever. So yes, in that sense, I guess she is the proverbial *one* …

I may have paraphrased, but only slightly. Our enthused American was in raptures, having been granted the secret to everlasting love, which apparently resides in a mathematical concept known to every maths undergraduate.

The encounter took place during my transition from the ivory towers of academia to the 'real world', where I would soon have to convince employers of my subject's relevance. As bemusing as the conversation was, it offered a clue on how to answer that oft-asked question concerning maths: *when will we ever need this*?

Mathematicians often find themselves on the defensive when society demands they explain how their work benefits others. When they do serve up a response, it usually takes one of three forms. The first, which is the most direct, draws on the practical benefits of maths in tackling real-world problems. Mastering percentages and fractions helps us to spot bargains (and rip-offs) at the supermarket. Engineers use trigonometry to design sturdy structures. A grasp of statistics equips us to analyse headline-grabbing claims. Probability gives us an edge in the stock market and helps us forecast election outcomes. The benefits of so-called 'applied maths' go on and, for many people, they provide the only justification maths needs.

But try telling that to so-called 'pure' mathematicians, who spend most of their time working on abstract concepts that are decidedly removed from real-world consideration. They liken maths to an art form and speak of the subject's aesthetic qualities.[1] They seek beautiful patterns and elegant arguments, with no regard for utility. Pure mathematicians often take pride in the apparent uselessness of their work, even deriding the supposed need for their subject to bring practical benefits. 'Here's to pure mathematics', starts one toast, 'may it never be of use to anyone.'[2] It is an elitist view – and one that I must confess to being endeared by as a student. It was only when I left academia, and my salary suddenly depended on the applicability of my skills, that I felt an impetus to justify the subject in more concrete terms.

The third response bridges these views by suggesting that even the most abstract and arcane mathematical ideas have a knack of finding their way to some application or another. Prime numbers, fractals and imaginary numbers all originated in the playground of pure mathematics but now underpin the world in which we live, from internet security (primes) to the design of cities (fractals) and the transmission of radio waves (imaginary numbers). This is what the American educator Abraham Flexner meant by the 'usefulness of useless knowledge' in his 1939 essay of the same title, when he argued that many of humankind's greatest scientific breakthroughs owe a huge debt to scientific inquiry that may initially be dismissed as pointless.[3]

My own answer to the question *'Why maths?'* is a variation on this theme. I have come to appreciate the subject as a collection of portable thinking tools that enrich the way we see the world, even when we do not expect them to. Mathematicians spend so much of their time in the abstract realm that the objects they study routinely seep into their ways of thinking. Our wise Brit, after all, did not learn the principle of induction with love in mind. Mathematicians study these concepts because they bolster their efforts to solve abstract problems – yet the more they mull over ideas in the mathematical universe, the more those ideas imprint themselves in their minds and influence their thoughts and behaviour.

Spend enough time with a mathematician and you'll uncover the telltale signs. They will speak with agonisingly precise vocabulary that reflects the rigour of their subject. Their conversations – even those that are not about mathematics – are filled with words like 'orthogonal' (irrelevant) and 'modulo' (except for). In their approach to everyday problems, mathematicians will naturally, even subconsciously, resort to the models they have acquired and embedded through years of study.

The usefulness of abstract maths should not come as a complete surprise. Many concepts derive in the first instance from a real-world context. Take the example of dimensionality. We are attuned to perceive the world in up to three physical dimensions and have a natural sense of what each one represents. Mathematics, as we shall see, gives us a simple and precise framework to describe objects in one, two or three dimensions. But before we know it, this framework has us making sense of four, five, even infinite-dimensional spaces. It's heady stuff, abstraction to its core. But the pay-off, aside from the interesting maths that arises, is that it gives us a more expansive way of thinking about space and, as Chapter 8 explores, it extends our understanding of various real-world phenomena beyond the usual three dimensions.

This book is a catalogue of mathematically inspired mental models – concepts that enrich our worldview and help us approach everyday situations. With a few exceptions, I have steered clear of more obvious examples. Well-trodden topics from statistics and related fields receive little attention here; there are plenty of excellent books that show us how to make sense of data. This book is focused on more abstract mathematical ideas that you probably will not have encountered before – or if you have, will show them in a new and unexpected light.

The majority of the mental models in this book are sourced from undergraduate courses. We will shed them of their technical baggage and focus only on the big-picture ideas they represent. If you are currently studying for a maths-related degree, consider them a preview of how abstraction will inevitably shape your understanding of the world. If you are a self-proclaimed mathophobe, do not be deterred – these models are eminently graspable, regardless of your prior experiences with the subject. Mathematicians benefit enormously from the thinking tools of their trade, and it turns out that everyone else can too with surprisingly little effort.

…

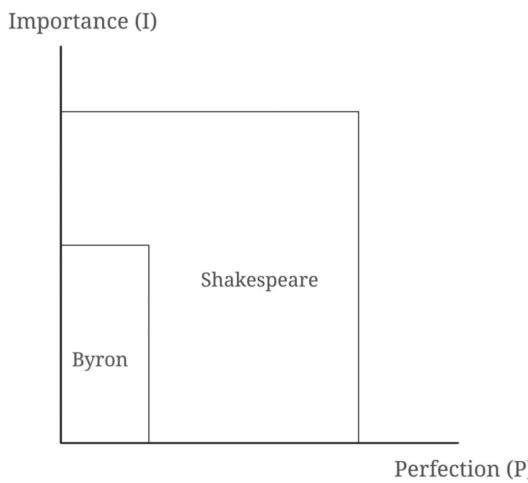
No mental model should be above reproach and you might already have taken issue with the Calm Brit's account of falling in love. At the time of that exchange in the common room, I was a bachelor in my mid-twenties, with no immediate prospect of love. It's fair to say

I was intrigued; maybe the same path lay ahead for me – an application of mathematical induction that would land me a life partner. But I was also sceptical. Were love reducible to logic, it would save the world from so much heartbreak. Mathematicians would have romantic partners queuing up (which, for the avoidance of doubt, they do not). Hollywood would rewrite its scripts, the mysteries of romance giving way to cold, calculated reasoning. Love, in all its maddening complexity, defies any serious attempt at mathematical description.

Attempts to 'mathematise' the world can undoubtedly go too far. There is a wonderful scene in the film *Dead Poets Society* when schoolteacher John Keating, played by Robin Williams, asks a student to read a passage from a textbook that references the Pritchard Scale of Understanding Poetry, in which the supposed 'greatness' of a poem can be quantified with the formula

$$\text{Greatness} = \text{Perfection} \times \text{Importance}$$
$$(\text{or } G = P \times I, \text{ in shorthand})$$

A sonnet by Byron, according to the passage, scores high for I and moderately for P, whereas one by Shakespeare scores high on both. By placing P on a horizontal axis and I on a vertical axis, we can visualise a poem's overall quality in terms of the area under the rectangular bar it forms, with Shakespeare occupying a greater area than Byron:



As his students diligently copy out the passage, Mr Keating gives a one-word verdict on the approach, 'excrement', instructing the students to tear the page from their books. The sciences are necessary to sustain life, Mr Keating explains, but pursuits like poetry are what we stay alive for.

Taking my cue from Mr Keating, I will declare outright that this book does not contain a prescription for 'the good life', whatever that means. That's not the point of mental models. Rather, they should illuminate our ways of thinking – bad as well as good. The Calm Brit's choice of model – induction – rests on the questionable assumption that stable patterns will persist in the future. The assumption often holds up in the realm of numbers, but it's an example of abstraction gone too far: the real world is a little too messy, our lives a little too unpredictable, than his argument implies. We'll return to induction in Chapter 6, by which point we will have encountered a host of other models that go some way to explaining where the Calm Brit falls short. None will quite crack the code for love, but they might offer a new perspective on how relationships develop. The reason induction has made it into this book is precisely because it exposes a flaw in our everyday thinking – not just in matters of love but in how we cling to the orderliness of our daily routines and refuse to countenance how they might be uprooted at any moment.

The models in this book are simply lenses through which to see the world. A few you will adopt right away because they bring clarity and depth to your view of a particular situation. Others will help you to understand how human thinking can go awry. Some will feel like folk wisdom because a lot of mathematical concepts are just formalisations of things we know to be blindingly obvious, but that we sometimes need reminding of in a world awash with misinformation and bad advice. There are reflections on productivity and parenting, on politics and sport, on relationships and on the idiosyncrasies that fill our lives. Some of the models are intended to be serious, others light-hearted. All of them are a matter of perspective. At the very least, if you ever find yourself surrounded by caffeine-fuelled mathematicians, you might just find yourself able to make sense of their musings.

So here's to pure maths – may we all benefit from its usefulness. And here's to you for daring to think like a mathematician.

1

The Continuum

The infinitesimal detail of the everyday

Several years ago, I went to a doctor seeking help for a chronic stomach condition that causes bouts of severe pain. I was asked to mark my discomfort on a five-point chart of emojis. The leftmost emoji was a picture of serenity, very different from the anguished-looking emoji on the far right, with the ones in between representing moderate levels of discomfort. I was dumbstruck by how, for all the marvels of modern medicine, my pain assessment boiled down to this most blunt of instruments.

You might have felt similarly restricted when asked to rate your hotel stay, or your Uber ride or, dare I suggest, the book you are reading. Five-star rating systems are very much the norm, but your choices are limited to a handful of whole numbers; values in between are strictly off limits. While responses are averaged into a more precise value (usually given to one or two decimal places), for the individual rater the options can feel inadequate. The ride was smooth but not perfect – four stars would be too harsh, five stars too generous. What's more, with the pressure exerted on gig economy workers – where, for instance, Uber drivers are required to maintain ratings upwards of 4.6 – the rating scale becomes distorted. A rating of five stars is now expected, and even a four-star rating is viewed as an indictment of one's work and afflicts reputational damage. Feedback is reduced to a perfection-or-bust logic.

Social media content is often liable to the same issue – the 'heart' icon on sites like Instagram and TikTok, for instance, enforces the most dichotomous of responses to each post (where not clicking the heart is taken as disapproval or disinterest), allowing no room for an individual to register a nuanced opinion. The primary purpose of these buttons (beyond serving as a measurable unit of engagement for marketers) is not for us to reflect on the content in any meaningful way, but to help shape our algorithmically optimised feeds and to present us with more of the same. Adding to the problem, the comments section typically reinforces the most commonly held views in either direction. This problem is magnified on Reddit, a platform that is ostensibly designed for discussion, where users have the option of either upvoting or downvoting a post, as well as its comments. This system, which at first appears equitable, penalises unpopular opinions because downvoted comments are quickly concealed from view and collapsed into an ignominious section at the bottom of the post. Genuine debate is rendered almost impossible.

Discrete rating scales persist in spite of a readily available alternative: the sliding scale. It's a startingly simple idea: instead of deciding whether you like or dislike a post, the question now becomes one of degrees, the *extent* to which it swayed your opinion in either direction. Instead of pinning your Uber driver to one of five ratings you would have license to place them anywhere on the scale from 1 to 5, whole number or otherwise. And fluid concepts such as consciousness could be viewed not in terms of binary on/off labels but an entire progression of states between those two extremes.

The sliding scale gets overlooked because it requires more mental effort on the part of the rater. Like many facets of online decision-making, feedback systems are designed to reduce our cognitive burden and to mindlessly push us towards quick and simple actions. Limiting our choices to predefined marks on a scale is akin to thinking the rainbow is made up of just the seven colours of ROYGBIV we learned in school. Those colours represent specific wavelengths (in increasing order) but we could just as well reference ten colours, or a hundred, or indeed any number between the two extremes of red and violet. Isaac Newton, whose experiments led to the discovery of the visible light spectrum, attached mystical significance to the number seven, which is probably why he settled on that many markers. But the clue is in the name: the visible light *spectrum* is also a sliding scale, an endless stream of colours situated between red and violet that our eyes readily detect.

This chapter makes the case for the sliding scale – or, as mathematicians call it, the *continuum*. From a mathematical standpoint, a continuum is simply an unbroken line of numbers. That's *a lot* of numbers – infinitely many, of course. Yet even this description doesn't do it justice. Mathematicians have uncovered a surprising truth about numbers that reveals just how vast the number line is and, thus, how much more expansive the continuum is as a mental model compared to its discrete proxies.

'God made the natural numbers, all else is the work of man'*

Back to the doctor who, sensing my discomfort at matching my pain level to an emoji, followed up with a wink and a nod: 'Look, Junaid,' he said, 'I know you're a numbers guy, so let's ditch the faces and think of it as a scale from 1 to 10. And you don't even need to give me a whole number. Use a fraction if you prefer!'

I appreciated the gesture, which in practice gave me ample options. Yet the purist in me wanted to tell the good doctor that even allowing for fractions, my options

* This quote is attributed to the nineteenth-century mathematician Leopold Kronecker.

remained paltry. Fractions are not all there is on the number line – in fact, they barely feature at all compared to other number types.

Think about all the numbers you know, and the order in which you encountered them. First came the whole numbers – tangible quantities like 1, 2, 3 – what mathematicians call 'natural numbers'. Your number line soon extended to accommodate zero, and also the negative numbers (–1, –2, –3, etc.) that head in the opposite direction.

As you learned to divide numbers, you realised that a new type of number was needed, because, for instance, dividing 6 by 9 does not leave you with a whole number. Between 0 and 1 alone we can slot in numerous fractions: we can slice that segment into halves, or thirds, or quarters. We could, if we so desired, slice it into seventy-sixths, marking ticks at each new point. The fifth tick along represents the fraction $\frac{5}{76}$, the seventeenth tick marks $\frac{17}{76}$. And so the number line, which accommodates every one of these conceivable fractions, suddenly feels very concentrated indeed.

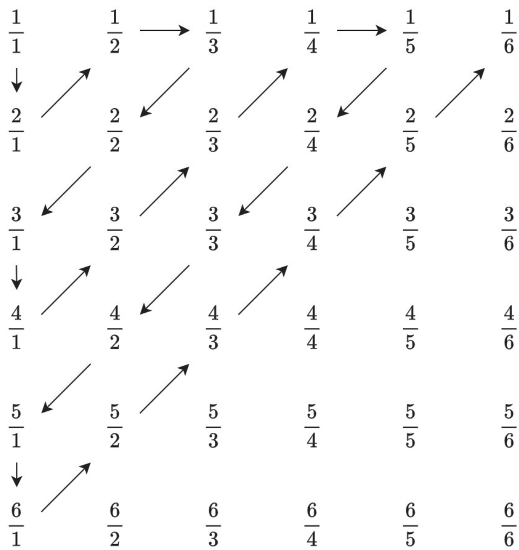
It's tempting to think that's our lot – that the number line comprises *only* the fractions. This was the conventional view among such luminaries as the Greek mathematician Pythagoras, for whom all quantities in the universe could be thought of in terms of a proportion. In this view of the mathematical world (a view apparently adopted by my doctor), all numbers are fractions.

But it turns out there are some numbers that cannot be expressed as fractions, however hard we try. The best known is everyone's favourite constant, pi (denoted by the Greek symbol π), the ratio of any circle's circumference to its diameter. This may sound like a fraction, but there are no two *whole numbers* whose ratio is exactly π. One of the main characteristics of these so-called 'irrational numbers' is that their decimal expansion will never repeat or terminate. The usual approximations of π, such as 3.14, are easy enough to pin down because they are expressible as fractions (in this case, just slice the interval between 3 and 4 into 100 equally sized smaller intervals and find the fourteenth tick along). But to capture a number like π in all its glory, it would not suffice to carve your number line into whole numbers, or hundredths, or even thousandths or millionths. You would have to carve up the number line repeatedly, infinitely. This hints at the incredible scope of the continuum.

So every number is one of two types: a fraction or an irrational number. Between any two numbers, you can find an infinite number of fractions and an infinite number of irrational numbers. Perhaps our comparisons should end there, but to truly grasp the extent of the continuum we must go further. One of these infinities, as we'll see, is larger than the other.

Our intuitions might lean towards thinking there are more fractions than irrationals because, as proportions, they seem more tangible than oddities such as π. And for practical real-world problems, we never need all those infinite digits anyway – NASA engineers do just fine with fifteen digits of π. The Pythagoreans could not countenance that such numbers could even exist; in their defence, the irrational numbers do not exactly jump out at us. The fractions must surely be the dominant force on the number line. Yet the exact opposite turns out to be true. Fractions might seem incredibly populous, but they can be placed in a single list. There's no way to order them in size, but they can be listed in a way that leaves no one out – they are said to be *countable* (an example of where mathematicians seem to have dropped the ball on terminology; *listable* would make more sense).

The method for listing the fractions is as simple as it is clever: we can first generate a grid of all possible fractions by making the rows correspond to the numerator (the fraction's top number) and the columns the denominator (the fraction's bottom number). The fraction in the sixth row and eleventh column, for example, is $\frac{6}{11}$. We have all possible fractions (some of them multiple times, in fact) in our grid. If we move diagonally through the grid, starting in the top left corner, we tick off every item in the grid – not a single fraction is missed.*



The beginning of the list of fractions (excluding duplicate values) is:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4} \ldots$$

Can we do the same with all *numbers*? Suppose I give you a list that I claim contains all the numbers. It will be

* Strictly speaking, this method only covers the case of positive fractions. But once they are shown to be countable, it is straightforward enough to modify the list to also include every negative fraction (simply place each one after its positive counterpart).

infinitely long, of course. Perhaps the first few numbers would look something like this, where each number is written as a decimal.

7.628423 …
0.500000 …
3.141592 …

The … denotes the fact that these expansions carry on forever. In the case of the second item, it is a never-ending sequence of 0s (we could just leave out the 0s, as they don't affect the value of our number, but you'll see why I'm including them shortly). The digits of the third item, which happens to be the number π, never repeat in this way.

If my list is the genuine article, I'll have shown that the whole number line is countable, just like the fractions. But if my list is not comprehensive, we should be able to find a number that doesn't appear on it. And it turns out we can indeed construct such a number, by dodging every item in my list as follows:

- The first digit is anything other than the first digit of the first number in my list – that is, anything other than 7.
- The second digit is anything other than the second digit of the second number in my list – that is, anything but 5.

- The third digit is anything other than the third digit of the third number in my list – that is, anything but 4.
- And so on.

As this is a number whose decimal representation disagrees with every number on my list (it will disagree with the eighteenth number on my list in the eighteenth digit and with the hundredth number on my list in the hundredth digit, for instance), it cannot therefore be on my list. Any attempt to construct an all-encompassing list of numbers will fail in much the same manner.

So the collection of all numbers is not countable. The fractions are not to blame – they are countable, remember. The problem lies with the irrational numbers; as unnatural as they may seem, there are simply too many of them to enumerate in a list. The extent to which they outnumber the fractions cannot be overstated – if you throw a dart randomly at the number line, you are not just *likely* to hit an irrational number; it is a mathematical certainty.

Let's take stock: in sizing up numbers, we have landed on four distinct sizes. Between 0 and 1 alone, there are:

- Two whole numbers – 0 and 1.
- Regularly spaced points such as tenths, hundredths or thousandths. Whatever our

choice of spacing, the number of points remains finite.

- All fractions – an infinite collection that can be enumerated in a list.
- All numbers (fractions and irrationals combined) – still infinitely many, but now they cannot be placed in a single list.

In quantifying just how 'many' numbers the number line plays host to, we've unravelled a new level of infinity – the collection of these numbers is 'uncountable'. There's nothing special about the segment from 0 to 1; any segment of the number line, however large or small, is a continuum. The whole number line (stretching forever in both positive and negative directions) is a continuum, but so is a segment between 0 and one-trillionth. That's largely what makes the continuum so vast; however much you zoom in, however tiny the segment, you still have uncountably many numbers at hand.

These four levels bring increasing degrees of resolution to our thinking. Let's take the previously mentioned example of consciousness. The first level represents the binary views of it as an all-or-nothing phenomenon where we apparently switch between conscious and unconscious states. But consciousness takes many forms such as daydreaming, drowsiness, hypnosis and sleep, each reflecting a different degree to which we are awake and alert to our surroundings. It's unlikely that consciousness can be boiled down to any finite number of states (the second level) or even a countable list of states (the third level). Since we appear to move fluidly between different levels of alertness, there is a strong case for thinking of consciousness as being situated on a continuum (the fourth level).[1]

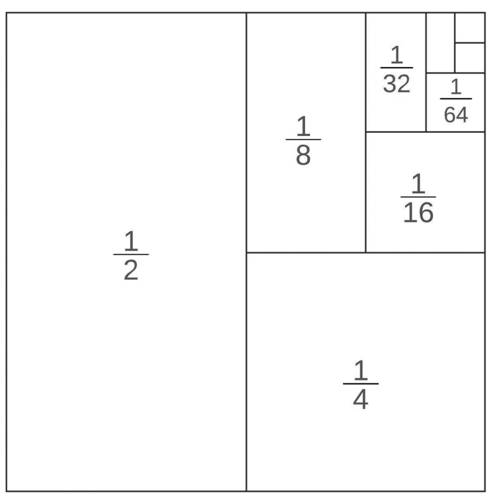
Another distinguishing feature of the continuum that deserves brief mention is that it contains no gaps. The fractions do have gaps: between any two fractions there are many numbers that are not themselves a fraction. But between any two numbers on a continuum, every intervening number remains on the continuum – there are no sudden breaks.

This conforms to our real-world intuition. According to classical physics, motion is continuous: as I'm running, my centre of mass is moving through a continuum of points in space. Time is continuous too: even though we break the year into 365 days, and each day into 24 hours, time progresses through a continuum of moments.

This might seem an obvious point, but it eluded many of history's smartest minds. The Greek philosopher Zeno had no concept of a continuum, which led him to the paradox that bears his name. There are several variations of the paradox; the most straightforward observes that to travel, say, one mile, you must first travel half a mile, then half of what remains, and again, and on and on. Since there is always some distance remaining, Zeno's logic dictates that you will never reach the end – time and motion are but an illusion.

The continuum refutes Zeno's logic by permitting us to break the journey into an infinite number of steps, each one halving in size: half a mile, then a quarter of a mile, then an eighth of a mile and so on. So the total journey is given by summing these terms:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1$$



The infinite sum visualised with segments of a 1 × 1 square. Each fraction corresponds to the area of the smaller segment it is in. The sum of all (infinite) of these fractions is the area of the 1 × 1 square, which is just 1.

Never has an ellipsis carried such significance: this one means we keep summing the terms, indefinitely. It is an act of imagination that resolves our conundrum because we can see that this infinite process results in a total of one mile. There's an infinite number of terms to sum, but they are shrinking sufficiently quickly for the sum to *converge* on a value. The size of the steps halves each time, approaching zero while never quite reaching it. The steps can be thought of as an ever-diminishing line that increasingly resemble a single point, without ever quite becoming one. Zeno's blind spot arose only because he lacked the tools to apply a simple process – that of adding ever-smaller distances – over and over again, infinitely many times.

The core of calculus

The continuum takes centre stage in one of the most powerful areas in mathematics: *calculus*. At its core, calculus is the study of change – such as how an object transforms from one state to another, or how the trajectory of a ball changes as it moves through the air. The change may be imperceptible, but the continuum can cope with this because it can be broken into as many parts as we require.

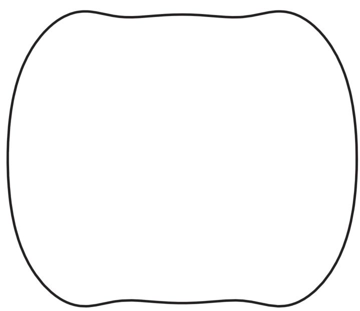
It's a shame Zeno never got to meet Usain Bolt because the sprinter, perhaps more than anyone in recent human history, puts to bed the notion that motion is illusory. We can put a number to Bolt's exploits using the formula we learned at school: speed = $\frac{\text{distance}}{\text{time}}$. In his world-record-breaking run, Bolt covered 100 metres in 9.58 seconds, a speed of 10.44 metres per second (or 23.49 miles per hour). This estimate is crude, however, because it assumes a constant speed throughout. If we want to measure Bolt's speed in his absolute pomp, it would be better to ignore the early and latter stages of the run, where he is not at full speed. We might instead examine his speed at the halfway point, 50 metres in. If we consider the 40-to-60 metre range, we have a distance of 20 metres and it takes Bolt 1.67 seconds to cover that interval, for a speed of $\frac{20}{1.67} = 11.98$ metres per second (or 26.95 miles per hour).

Our measure of Bolt's speed at the 50-metre mark becomes sharper as we shrink the interval around it. Perhaps 5 metres either side of the 50-metre mark. Or just 1 metre. Or half a metre. Or less than that. And so on, by which I mean indefinitely. As we go through each iteration, the interval narrows towards a single point at 50 metres, without ever quite reaching it. Time approaches zero, but never quite reaches it. And Bolt's actual speed at the 50-metre mark is whatever the estimates converge towards, their so-called *limit*.

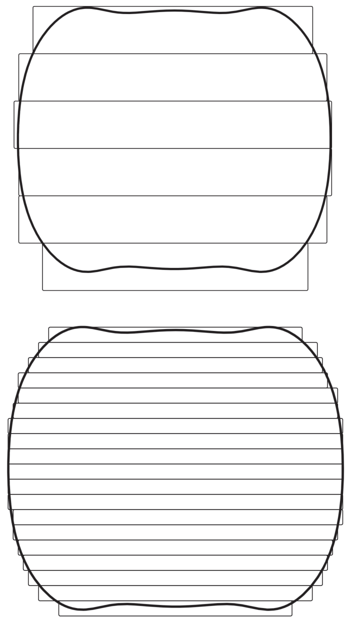
That, in a nutshell, is the basic idea of calculus: to break measurement into *infinitesimally small* parts.

We have effectively performed the calculation $\frac{0}{0}$, usually forbidden in maths, by approximating both the top and the bottom by values that approach zero, without ever quite getting there.

This method is routinely called on by mathematicians to measure the area of all manner of irregular shapes for which we don't have neat formulae. Suppose we want to estimate the area of this awkward-looking shape:



One way is to fill it with rectangles. We can easily compute the area of each rectangle and then find the sum. This won't give us the precise answer, of course; the rectangles stray over the shape's boundary, meaning we'll end up with an overestimate. We can do better by reducing the width of the rectangles. The process will be the same, but we'll have more rectangles and a better approximation of our shape's area. And, as you might have come to expect, we can continue this process forever, with the rectangles converging towards lines that collectively fill the space inside our shape and nothing else. Once again,



Two estimates of the area of our irregular shape. As the rectangles become thinner, the estimate becomes more accurate.

we never actually reach this point – the rectangles never actually become lines – but we can allow the process to go on and see where it leads (in this case, towards a precise answer for the area of our shape).

Invoking the continuum reflects a pointed approach to problem-solving that the mathematician and writer Steven Strogatz likens to an extreme form of divide-and-conquer. 'All good problem-solvers know that hard problems become easier when they're split into chunks', says Strogratz.[2] 'The truly radical and distinctive move of