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The Uncertain Science of Certainty

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For O & R

Contents

INTRODUCTION

In April 2010, 6 million passengers were left stranded around the world after the Eyjafjallajökull volcano erupted in Iceland. I was one of them. On Friday, 16 April, I had been due to catch a flight across the Atlantic back to London, via Amsterdam. But the day before, countries in northern Europe had started closing airspace after detecting a 30,000-foot-high volcanic ash cloud heading their way.

Millions of minds were suddenly united in a single question: when would things reopen again? Would it be days? Weeks? Or more? According to the *Guardian*, 'one volcanologist said the ash could present intermittent problems to air traffic for six months if the eruption continued'.[1]

Historically, planes and volcanoes had not mixed well. If ash particles get pulled into aircraft engines, they can melt, clogging up the blades and vents. In 1982, a British Airways 747 had lost all four engines after flying through an ash cloud near Jakarta; the same had happened to a KLM flight over Alaska in 1989. (Fortunately, both planes managed to restart their engines after the ash cooled then broke away in glass-like fragments.) Subsequent aviation guidelines therefore included a new recommendation: planes should entirely avoid airspace containing volcanic ash.[2]

But it was unclear whether the Eyjafjallajökull ash spreading across Europe would have the same effect as the dense clouds in previous incidents. Worse, there wasn't even good data on where the ash cloud was. With flights grounded, tensions rose as different groups tried to make sense of the available evidence. Governments wanted proof it was safe to fly. Airlines wanted proof that it wasn't.[3]

The UK, which was closer to the cloud than many other countries and had a long history of climate analysis, initially adopted a cautious approach. Researchers used atmospheric simulations to predict the spread of ash, building on methods that had first been developed to track radioactive particles from the 1986 Chernobyl nuclear accident. Even though subsequent data would be consistent with the predictions, knowing the location of the ash cloud was not enough. Nobody could be certain what the ash meant in terms of actual risk to aircraft.

While some groups ran simulations, others decided to run experiments. On Saturday, 17 April, KLM – the airline I happened to be booked with – sent up a test flight without passengers. Flying over the Netherlands, everything went smoothly and they found no evidence of damage afterwards.[4] More test flights followed, by more airlines. As data accumulated, and the perceived risk declined, passenger flights gradually started to resume.

The ash cloud would move on, but deeper questions remained. Faced with a new problem, how should we weigh up existing theories and emerging observations? When is it ethical to gather new data if experiments come with a risk? And where should we set the bar for making a decision?

Proof often carries a certain urgency. It is not just about what is true; it is about convincing ourselves – and others – that something is true. If prosecutors want to stop a murderer walking free, they need to assemble enough proof to persuade a jury. If countries want to roll out a new vaccine during an epidemic, they need proof it's sufficiently safe and effective. If we want to have fewer legal errors, or more medical advances, or make better decisions about whether to send up planes after a volcanic eruption, we need to seek out proof.

Yet our experience of proof can often feel fleeting and forced: here's an established fact, remember it, recite it. In the early stages of my mathematics degree, it was drilled into us that we should never start a statement with 'Clearly' or 'Obviously'. If our logical steps are convincing, the reader should find them clear without us saying so. I'd later discover that writers have a similar mantra: 'show, don't tell.' But that still leaves us with the problem of choosing what to show, and how to show it.

Approaches to proof often depend on the situation we face. How would you show a new product is better than the old one? Or persuade a court of someone's guilt? What would make you trust a self-driving car, or a financial transaction with a stranger? And which types of evidence would you want for an incoming government policy, or – if you were to be ambitious – the founding principles of a new country? Would you approach these questions like a philosopher, a lawyer, a computer scientist, or a statistician? Or someone else entirely?

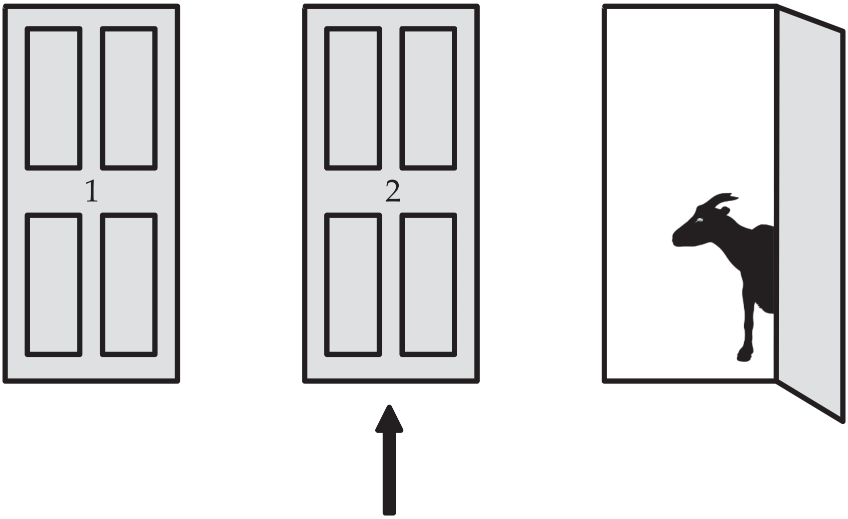
Life is full of situations that can reveal remarkably large gaps in our understanding of what is true and why it's true. This is a book about those gaps. It is the story of the ideas that have helped scientists and societies discern between truth and falsehoods, improving decision-making and reducing dangerous errors. From medieval juries to modern scientific revolutions, it is about the methods people have used to accumulate evidence, negotiate uncertainty and converge on proof. And, crucially, what happens when those methods fail.

Furious letters flooded in from readers with PhDs. The target of their doctoral-level anger? Marilyn vos Savant, who held the record for the highest IQ in the world. It was September 1990, and vos Savant had just written a magazine column about a seemingly innocuous mathematical puzzle. Unfortunately for her mailbox, the solution she'd provided would cause an uproar. 'May I suggest that you obtain and refer to a standard textbook on probability before you try to answer a question of this type again?' wrote one unhappy reader. 'How many irate mathematicians are needed to get you to change your mind?' demanded another.[5]

There would be thousands of letters in total, several of them on university letterheads. Many were patronising: 'Our math department had a good, self-righteous laugh at your expense.' Some went further, lamenting the harm she had caused to education: 'There is enough mathematical illiteracy in this country, and we don't need the world's highest IQ propagating more. Shame!' Others decided the error lay in biology: 'Maybe women look at math problems differently than men.'

The puzzle that had sparked such a furore was commonly known as the 'Monty Hall problem', after the US game-show host, and had been around since the 1970s, even though many mathematicians were evidently still unfamiliar with it. I would first hear about it as a teenager, a decade after the vos Savant controversy. But it would take me far longer to properly appreciate the puzzle's implications.

If you haven't come across the Monty Hall problem before, the premise is simple. You're on a game show and



If you choose door #2, then the host reveals a goat behind door #3, would you switch?[6]

must choose between three doors. Two doors have a goat behind them and the other a luxury car. You start by picking a door, say door #2. Then the host, who knows what's behind the doors, opens one of the other doors, say door #3, to reveal a goat. He then gives you the option to switch your choice and pick door #1 instead. Should you take up their offer?

In 1995, mathematician Andrew Vázsonyi told his friend Paul Erdős about the puzzle.[7] If anyone could work it out, it should be Erdős, who'd published more mathematical papers than anyone in the world and was specialist in probability to boot. Would he switch, given the choice? Erdős said that 'it should make no difference'. There were two doors left, he reasoned, so it should be a 50/50 gamble.

His response surprised Vázsonyi, who'd spent time working through all the combinations of possible choices, eventually deciding it did make a difference. For example, suppose the car is behind door #1 and you correctly choose this door at first. If you later switch, you'll lose the car. But now imagine you'd picked door #2 instead. Remember that the host reveals a goat behind a door that's not the one you chose. So in this scenario, it means the host opens door #3. As a result, switching would win you the car. Likewise, if you'd picked door #3 and the host reveals a goat behind door #2, you'd win by switching. In other words, two times out of three you'll get the car if you switch. The same logic holds if you think about what would happen if the car were behind another door initially. Vos Savant had given the same conclusion in her column: mathematically speaking, it's better to switch.

When Vázsonyi ran Erdős through his calculation, though, he got a second surprise. 'To my amazement this didn't convince him,' Vázsonyi recalled. 'He wanted a straightforward explanation.' This stumped Vázsonyi, who couldn't think up an intuitive reason for his solution. Trudging through all the possibilities had shown him what he should do, but not *why*. Later that day, Erdős asked him again. 'What's the matter with you? Why aren't you telling me the reason why I should switch?'

Still unable to come up with an answer, Vázsonyi decided to take an alternative approach. He put together a computer simulation of the game and ran it 100,000 times. The results showed that if he chose to switch every time, then he won the car in two thirds of the simulations. If he stuck every time, he won the car in only a third of them. To his relief, his friend finally relented. 'Erdős objected that he still did not understand the reason why, but was reluctantly convinced that I was right.'

Although Erdős – and vos Savant's angry readers – had been wrong, it wasn't just a matter of failing to land on the correct answer at first. Even when someone had led them right to the solution, they still struggled to mentally navigate their own way there. For many, the Monty Hall problem causes lingering frustration. Against my better judgement, I've sometimes been drawn into long conversations about the riddle when people hear about my mathematical background. Like Erdős, they wanted an intuitive explanation, a neat conceptual map that can guide them to the correct destination. And that's where, like Vázsonyi, I've often struggled. When I first encountered the puzzle, I was soon convinced of the solution. But I didn't realise how tricky it would be to convince others. Even if you've heard of the Monty Hall problem before, and remembered the correct answer when you saw the doors above, how confident are you that you'd have been able to give others a satisfactory explanation? If put on the spot, could you move beyond a recited solution and into the deeper realms of proof?

Even if it takes some time to intuitively grasp the solution, Vázsonyi showed that it's still possible to work out the correct approach by simply playing the game repeatedly. As it happens, this seems to be the preferred method of some non-human species. A series of experiments published between 2010 and 2012 found that pigeons could often work out the optimal Monty Hall strategy through trial and error, especially if they were given an extreme version of the game with a larger benefit to switching.[8] Even so, a year after her original column, and despite several follow-up explanations, several humans were still writing letters to Marilyn vos Savant. 'I still think you're wrong,' one man told her. 'There is such a thing as female logic.'

The Monty Hall problem shows that even apparently simple problems can be stubbornly divisive. In his conversations with Erdős, Vázsonyi tried two different approaches to prove his solution was correct. When he ran through all the potential car locations and choices of door, recording each result systematically, he was employing a *proof by exhaustion*. Every combination was tested, every possibility exhausted. In contrast, when he used a computer to play the game thousands of times, and tally up the outcomes, it was a *proof by simulation*. Strategies could be tested again and again, with the best one emerging through a large volume of attempts.

In a story recounted by his contemporaries, the philosopher Socrates distinguished between knowing something is true and merely believing something that happens to be true by our ability to explain why.[9] Without explanation, we can end up like Erdős, unsatisfied but reluctantly accepting. To avoid such shortcomings, we must look at how different forms of proof have succeeded and stuttered. Over the coming chapters we'll explore a variety of approaches, from the logical elegance of *proof by contradiction* and *proof by contraposition* to the controversies of *proof by construction* and *probabilistic proof*. We'll unpick the complexities of *proof beyond reasonable doubt* and the damaging effects of *proof by assertion* and *proof by intimidation*. We'll trace the evolution of scientific thinking, from the quirks of ancient logic to the trustworthiness of modern artificial intelligence. And we'll see how these ideas have shaped our concepts of politics, justice and risk.

We will also encounter the limits of proof. Although Marilyn vos Savant held the Guinness World Record for highest IQ between 1985 and 1989, she readily admitted that this didn't mean she was the most intelligent person out there. 'So many factors are involved that attempts to measure it are useless,' she once said.[10] In her view, IQ tests were inevitably limited in what they could conclude. 'They measure a variety of mental abilities, and the best tests do it well. But they don't measure intelligence itself.' As it happened, the *Guinness Book of Records* would 'rest' the highest IQ category in 1990, the year of vos Savant's Monty Hall column.[11] Given the range of tests available, and variation in results, it had been decided that it didn't make sense to try to crown a single record holder.

Throughout history, the emergence of truth has relied on people finding effective methods of analysis. But acceptance of proof can depend on social factors too. When you first read Paul Erdős's answer to the Monty Hall problem, were you tempted to believe it because of who he was? Or did you share the motto favoured by the scientists of the UK's Royal Society: *Nullius in verba* ('Take nobody's word for it')? From misinformation and conspiracy theories to paradigm shifts and scientific rifts, it's not just a matter of evidence, it's also about the social dynamics around that evidence.

Such issues have become increasingly prominent in recent years. Ten years after tiny ash particles from Eyjafjallajökull closed European airspace, the global spread of coronavirus particles would lead to even larger and longer shutdowns.[12] For many non-scientists, the COVID pandemic was a revelation about how science works. The inner workings of research, often fleeting and abstract in pre-pandemic life, dominated headlines for months on end. In real time, the public discovered the sometimes ugly reality of how the scientific sausages were made: the politics, the personalities, the prejudices.

Yet it was also an opportunity to stand at the edge of knowledge, sharing the view as new discoveries emerged. As an epidemiologist during that period, I worked to untangle the features of the pandemic, from disease severity to the emergence of new variants. We combined methods and datasets, searching for the best tools to pin down what were often shifting, fragmented glimpses of the truth. Sometimes our conclusions about the disease would be tested within days; sometimes it would take months to see if we'd been on the mark.

As my email and Twitter notifications started to fill with abusive messages, I came to sympathise with Marilyn vos Savant. Even when multiple sources of evidence pointed in the same direction, there were still people who were convinced of the opposite. In many ways, those intense years were a microcosm of proof in wider life, a jumble of evidence, statistics, logic and human behaviour.

After receiving that avalanche of criticism in 1990, vos Savant said that 'math answers aren't determined by votes'.[13] The apparent purity and permanence of mathematics are undoubtedly alluring, in science and beyond. In an uncertain world, it is reassuring to think there is at least one field that can provide definitive answers. But this certainty can be an illusion, with even mathematical notions of proof not always as robust and politics-free as they might seem. So that is where we shall start. With a group of mathematicians, a group of politicians and a shared crisis that would engulf them both.

1

THE NATIONAL AXIOMS

The audience recoiled at the sheer strangeness of the man on stage. Was this really who they'd come to see? 'At first sight there was nothing impressive or imposing about him, except that his great stature singled him out from the crowd,' recalled a lawyer who was there. 'His clothes hung awkwardly on his giant frame; his face was of a dark pallor, without the slightest tinge of color; his seamed and rugged features bore the furrows of hardship and struggle; his deep set eyes looked sad and anxious.'[1]

Earlier, as 1,500 people crowded into the venue, away from the snowy air and slushy streets outside, one of the event's promoters had become nervous about their supposed star. 'His dress that night before a New York audience was the most unbecoming that a fiend's ingenuity could have devised for a tall, gaunt man: a black frock coat ill-setting and too short for him in the body, skirt and arms, a rolling, low collar disclosing his long thin, shrivelled throat, uncovered and exposed.'

But as the man started speaking that Monday evening, with a sharp, sometimes shrill voice ringing through the hall, the lawyer noticed a change in the atmosphere. 'With no attempt at ornament or rhetoric, without parade or pretense, he spoke straight to the point.' The man's words, punctuated by erratic gestures, sparked a flurry of excitement in the crowd. 'He held the vast meeting spell-bound,' noted a watching journalist, 'and as one by one his oddly expressed but trenchant and convincing arguments confirmed the accuracy and irrefragability of his political conclusions, the house broke out in wild and prolonged enthusiasm. I think I never saw an audience more thoroughly carried away by an orator.' The next day, a copy of the speech appeared in four different New York newspapers. The man's words – and the support they drew – would later be reproduced across the United States. Within a year, he would be elected president.

A couple of weeks after the New York event, Revd John Gulliver managed to strike up a conversation with the speaker, a politician by the name of Abraham Lincoln, as he caught a train. The previous evening, Gulliver had seen him talk in Norwich, Connecticut,[2] and it was one of the most extraordinary speeches he'd ever heard. Taking a seat next to Lincoln, Gulliver asked him how he managed to be so convincing: 'I want very much to know how you got this unusual power of "putting things".'[3]

Lincoln said he didn't have the benefit of a formal education. 'I never went to school more than six months in my life.' However, during his training as a lawyer, he'd come across a different way of thinking. 'In the course of my law-reading, I constantly came upon the word "demonstrate". I thought at first that I understood its meaning, but soon became satisfied that I did not.'[4]

Digging through dictionaries and reference books, Lincoln couldn't find a description that really made sense. 'I consulted Webster's Dictionary. That told of "certain proof", "proof beyond the possibility of doubt"; but I could form no idea what sort of proof that was.' Eventually he decided the only way to understand the concept was to put it into action.

'At last I said, "Lincoln, you can never make a lawyer if you do not understand what 'demonstrate' means" and I left my situation in Springfield, went home to my father's house, and stayed there till I could give any proposition in the six books of Euclid at sight.'

Those six volumes – the opening half of a thirteen-book collection known as *The Elements* – had originally been written by mathematician Euclid in ancient Greece around 300 BC. So how did a 2,000-year-old text come to have such a profound influence on American politics? At first glance, *The Elements* seems like a dreary technical tome. There's no background to motivate the work, no mention of people and no illustrative examples. With quintessential brevity, *The Elements* literally gets to the point in its opening sentence: 'A point is that which has position but not dimensions.'[5] And yet Euclid's work would become a landmark, an ancient wonder that would outlast even the towering lighthouse in his home city of Alexandria.

Euclid had created a system for knowledge, a way of constructing seemingly universal truths from fundamental principles. At the foundation of *The Elements* were definitions. Following that opening definition of 'a point', Euclid went on to define shapes and structures. A triangle, for example, was 'a figure formed of three straight lines joined end to end'. Euclid also specified self-evident axioms that needed no proof, such as 'the whole is greater than its part'.[6]

From these definitions and axioms, Euclid then demonstrated dozens of mathematical claims. The structured manner of such proofs indicated a logical route to objective, undeniable conclusions. As a result, *The Elements* would become a bestselling guide to finding truth through reason, second only to the Bible in printed editions.[7] It would find a central role in classical education, influencing Western thought for centuries to come. As well as inspiring mathematicians and philosophers, Euclid would shape the very foundations of democracy in Europe and the United States.

Prior to ancient Greek mathematics, the focus had fallen mostly on tangible questions. For example, one Babylonian tablet asks: 'I have eaten two thirds of my provisions: there is left 7. What was the original amount of my provisions?'[8] Meanwhile, the Egyptian Rhind papyrus, written around 1650 BC, was posing questions like: 'A quantity added to a quarter of that quantity becomes 15. What is the quantity?'[9]

Motivated by such questions, these societies explored early mathematical rules for problem-solving, such as methods of guessing and checking. People subsequently applied the techniques to situations ranging from building to bartering, but there was still an inherent limitation to the ideas. Although the Rhind papyrus opened by promising the 'entrance into the knowledge of all existing things and all obscure secrets', descriptions were often imprecise and hard to generalise to other situations.[10] In Babylonian and ancient Egyptian mathematics, a solution to a question rarely extended beyond that specific situation. There were no universal truths, no theorems and no proofs.

Despite this devotion to specific problems, there were some conceptual innovations along the way. The Babylonians were the first to come up with a positional number system, where the same symbol could represent different values depending on where it sat in the number. In modern mathematics, for example, the symbol '1' in the number 111 simultaneously represents 100, 10 and 1. This is far more efficient than a method in which every new number requires an additional symbol.

Imagine writing out the number 111 using the tally system, which has prehistoric origins.

Nowadays, we're used to positioning digits based on multiples of 10, a method that is Arabic in origin. In contrast, the Babylonians used a base-60 system, with the larger units made up of sixty smaller ones, much like a stopwatch showing hours, minutes and seconds. For societies interested in problem-solving, structuring a number system around 60 has a major benefit: there are many ways we can divide it up (i.e. by 2, 3, 4, 5, 6, 10, 12, 15, 20 or 30) and get a whole number out at the end. Like the problems on those Babylonians tablets, it reflects knowledge developed around a desire to solve practical problems. If you are trying to construct a building, you don't necessarily need a universal truth; you just need a building-sized truth that means your efforts won't collapse.

Yet there are benefits to generalisation. If something is true for a particular triangle, sphere, prism or pyramid, it's useful to know whether it holds for other ones too. Tackling each new problem from scratch can become inefficient, with solutions and facts left scattered rather than building towards larger knowledge. It can also lead to inconsistencies: the Babylonians settled on the formula $3r^2$ for the area of a circle with radius *r*, whereas the Egyptians used $16r^2/9$. Only when the ancient Greeks arrived did mathematicians converge on the true formula of πr^2. This is why Greek mathematics, and particularly the ideas compiled by Euclid, ended up being so influential.[11] They gave people the fundamental elements needed to move from narrow problems to broad proofs.

The promise of sturdy, generalisable truths was appealing not only to mathematicians. One of the earlier pioneers of Euclidean thinking in politics was the Englishman John Locke. Born in 1632, his childhood had been a time of turbulence.

King Charles I had increasingly bypassed Parliament, claiming he had a divine right to rule, which eventually led to a civil war that would play out over almost a decade.[12] Locke, whose father had fought against the Royalists in the war, would go on to develop a great interest in the concept of morality. How should societies decide what is right, and what rights people should have?

While forming his ideology, Locke took guidance from the logical structure of *The Elements*. Locke believed that the 'natural rights' of a society could be established in a similar manner: defining concepts, then using these definitions to establish statements. For example, Locke defined property as 'a right to any thing', and injustice as 'the invasion or violation of that right'. The statement 'where there is no property there is no injustice' can therefore be considered self-evident, as it follows directly from the definitions given within it.[13] Locke suggested this conclusion was 'as certain as any demonstration of Euclid'.

In his *Two Treatises of Government*, published in 1689, Locke noted that 'creatures of the same species and rank, promiscuously born to all the same advantages of nature, and the use of the same faculties, should also be equal one amongst another'. It was therefore evident to him that, in the absence of rules or laws, humanity's natural state was 'free, equal and independent'. He concluded that the aim of a government should be to preserve the natural rights of life, liberty and property.[14]

The Age of Enlightenment was rising, with reason and logic challenging orthodoxy and authority. Before the revolutionary era, absolute monarchs and the Church had dictated laws and justice. Enlightenment thinkers sought to change this tradition. Rather than notions being defined from above, they wanted to identify natural, objective laws.

In France, Voltaire argued that objective reasoning had the power to unify populations in a way that religious sects could not. 'Every sect, in whatever sphere, is the rallying-point of doubt and error … There are no sects in geometry; one does not speak of a Euclidian, an Archimedean. When the truth is evident, it is impossible for parties and factions to arise.'[15] When writer Charles le Beau claimed that, unlike earlier religions, Christianity had a concept of morality, Voltaire criticised this as absurd, citing ancient Greek and Roman philosophers' work on morals: 'There is but one morality, M. Le Beau, as there is but one geometry.' Just as societies worldwide could derive the same geometric theorems, Voltaire suggested they could all converge on the same system of morals. 'The Indian dyer, the Tartarian shepherd, and the English seaman, are acquainted with justice and injustice,' as he put it.

Later, in Germany, Enlightenment philosophers would turn their attention to the concept of beauty. Immanuel Kant suggested that everyone should agree on its definition, rather than treating it like a personal opinion. He asked his readers to imagine a man declaring an object to be beautiful. 'If he gives out anything as beautiful, he supposes in others the same satisfaction – he judges not merely for himself, but for every one, and speaks of beauty as if it were a property of things.'[16] Kant saw morals and aesthetics as fundamentally intertwined: two universal truths about what is 'better'.

Enlightenment thinking would flow across the Atlantic, into the pens of the American Founding Fathers. The US Declaration of Independence, drafted by Thomas Jefferson in 1776, began its second sentence with 'We hold these truths to be self-evident, that all men are created equal.' Or at least, those are the words that made it into the final version. A keen student of mathematics, Jefferson had mimicked Euclid's and

Locke's logical style in writing the first draft, from his statement that 'all men are created equal' to a list of 'inherent' rights including 'life, liberty and the pursuit of happiness'. But his rough initial version hadn't specified the opening truths to be self-evident. Rather, Jefferson had written that 'we hold these truths to be sacred and undeniable'. It was only after Jefferson had shown the draft to his colleague Benjamin Franklin that the crucial edit was made.[17]

Like Jefferson, Franklin realised the value of mathematical thinking in wider life. He would leave his grandson his French translation of *The Elements*, and once wrote that mathematical demonstrations render the mind 'capable of exact reasoning, and discerning truth from falsehood in all occurrences, even subjects not mathematical'.[18] Franklin brought this outlook to his editing of the Declaration. With heavy pen slashes, he cut the words 'sacred and undeniable' – with their implied reliance on religion – and replaced them with the scientific claim of 'self-evident'.

Lincoln was surveying a vast canvas of diagrams when his colleague William Herndon entered the office. It was the early 1850s, almost a decade before his New York speech, and the pair were lawyers on the Illinois circuit. Lincoln was so taken with the shapes on the thick paper sheets that he barely acknowledged Herndon as he walked in. Bottles of ink lay scattered on the table, alongside pencils, a compass, a ruler and a tall stack of fresh paper. Lincoln had apparently been there since the early hours, and would continue to work on his problem throughout the day.[19]

Eventually Lincoln had to visit the courthouse, and on his way out he revealed to Herndon what he was battling with. The scraps and sketches on his table were the result of an effort to 'square the circle', an ancient puzzle which involved drawing a square and a circle of equal area. Using his pencil, compass and ruler, Lincoln had hoped to convert the principles of Euclid into a solution. Herndon knew Lincoln had read several of Euclid's texts, carrying them in his saddlebags and studying by candlelight while they toured the circuit. Lincoln had been known to study until the early hours of the morning, somehow maintaining his concentration among the loud snoring of his lawyer roommates at the country inns. But it was not enough. 'For the better part of the succeeding two days he continued to sit there engrossed in that difficult if not undemonstrable proposition and labored, as I thought, almost to the point of exhaustion,' Herndon recalled. After hours of drawing and measuring, Lincoln had to admit defeat.

Lincoln's frustration in trying to make two things equal would eventually extend to another, much larger problem. The Declaration of Independence may have opened with the 'self-evident' truth that all men were equal, but the continuing existence of slavery in the US was at odds with such notions of equality. Just as Euclid had influenced the words of the Founding Fathers, Lincoln used mathematical logic to argue against the practice. 'If A can prove, however conclusively, that he may, of right, enslave B,' he pondered in a private 1854 essay, 'why may not B snatch the same argument, and prove equally, that he may enslave A?' Regardless of whether the right to own slaves was defined by colour, intellect or money, an enslaver could always by the same argument be enslaved by a supposed superior.

It was a textbook *proof by contradiction*, which Lincoln had seen Euclid employ many times in *The Elements*. The method works by taking the statement we want to prove is true – let's call this statement 'P' for short – then showing that if P is false, it leads to a pair of mutually contradictory conclusions. The logical structure is therefore as follows: if we assume P is false, and this implies that another statement Q is true *and* the converse of Q is true, then P must be true.

In the early pages of *The Elements*, Euclid used this method to demonstrate that if two angles in a triangle are equal, the lengths of the sides opposite them will be equal too. First, he assumed that the statement was false: one side was longer than the other. Then he cut off the excess length on this imaginary triangle so both sides were the same length. However, this would mean the smaller triangle with the length cut off ends up being a mirror image of the original, and hence identical in size. This would contradict the axiom that 'the whole is greater than the part'. Therefore, both sides must be equal in the original triangle.

Lincoln's proof followed the same structure. To show that person A could not legitimately enslave B, Lincoln first assumed that A could enslave B. This implied that a legitimate argument for creating slaves existed. B could therefore use this same argument to enslave A, which contradicted the original

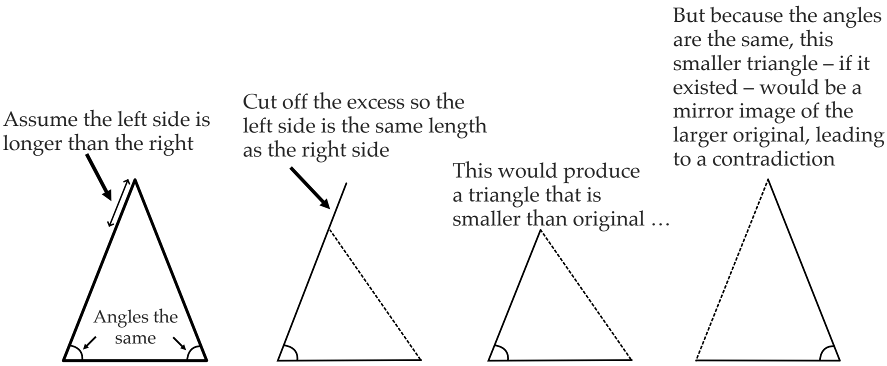


Illustration of Euclid's proof by contradiction of Proposition #6 in *The Elements*.

assumption that A was the enslaver of B. If one person had the right to enslave another, Lincoln concluded, then person A could be both enslaved and enslaver. Hence a person did *not* have the right to enslave another.

Unlike the drudgery of incremental mathematical calculations, a proof by contradiction offers the flourish of a high-stakes conclusion. Stumble upon the absurdity, and in that moment the entire proof falls into place. It was for this reason that the mathematician G. H. Hardy called proof by contradiction 'one of a mathematician's finest weapons'.[20] As he put it, 'It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game*.' And it was a technique that Lincoln would eventually deploy in one of the greatest contests of all.

Lincoln's political journey had encountered a bumpy start. In 1846, he'd been elected to the House of Representatives, representing Illinois's 7th district. But his vocal criticism of an ongoing war with Mexico had provided ammunition to his opponents, who'd claimed he was undermining the soldiers who'd gone to fight. As one journalist wrote, 'he has by his vote stigmatized the brave men of his state'.[21] Lincoln had briefly considered running for re-election, despite previously saying he would serve only a single two-year term, but the public reaction persuaded him otherwise. 'He told me himself that he felt by his course in Congress he had committed political suicide,' Herndon later recalled.[22]

Lincoln's subsequent return to life as prairie lawyer came with a new outlook. 'I could notice a difference in Lincoln's movement as a lawyer from this time forward,' Herndon noted. 'He had begun to realize a certain lack of discipline

– a want of mental training and method.' Those late nights studying Euclid's propositions were part of an effort to fix this deficit. Lincoln pondered and proved, sharpening his logic as well as his pencils. By the mid-1850s, he was ready to go back into politics. This time it would be different.

Lincoln's main rival on the Illinois political circuit during that period was Stephen Douglas, Democrat senator and former judge. The pair formed a stark contrast: Lincoln a foot taller than Douglas, his shrill voice contrasting with Douglas's booming speeches. They differed in their outlooks too. While Lincoln often travelled unaccompanied on public railways, striking up conversations with people such as Revd Gulliver, Douglas had his own private railroad carriage, complete with whiskey and other refreshments.[23] Douglas also opposed Lincoln's views on slavery. When the Founding Fathers talked of equality, Douglas claimed in 1857, they were not really referring to all men, 'they were speaking of British subjects on this continent being equal to British subjects born and residing in Great Britain'.[24] In a subsequent speech, Lincoln quoted Douglas and countered by following the logic to its absurd conclusion. If the Declaration referred only to British subjects, Lincoln argued, 'the French, Germans and other white people of the world are all gone to pot along with the Judge's inferior races'.

During 1858, Lincoln and Douglas – who were running against one another for the US Senate – engaged in a series of seven debates, with thousands turning out to watch each one. Once again, Lincoln turned to Euclid in forming his arguments. When Douglas criticised a legal point made by Senator Lyman Trumbull by calling him dishonest, Lincoln told Douglas that logical arguments could not be defeated with personal insults. 'If you have ever studied geometry, you remember that by a course of reasoning, Euclid proves that all the angles in a triangle are equal to two right angles,' Lincoln said. 'Euclid has shown you how to work it out. Now, if you undertake to disprove that proposition, and to show that it is erroneous, would you prove it to be false by calling Euclid a liar?'[25]

The proposition Lincoln had quoted was the thirty-second to appear in *The Elements*. Unlike earlier, self-contained proofs, this proposition relied on results that had already been demonstrated in three earlier propositions.[26] In turn, these depended on four previous proofs in *The Elements*, which were themselves based on earlier results. Lincoln's arguments would develop cumulatively as well, with one idea building on another. However, this approach could only work if the foundations of the argument – from its axioms upwards – remained solid.

The Lincoln–Douglas debates focused heavily on slavery, and who should decide its future in the country. Douglas argued that slavery was a sovereign right, to be decided by the people of each state. Lincoln disagreed: the notion of popular sovereignty should not include slavery. The following year, Lincoln challenged Douglas to construct a logically sound argument to support his case. 'If Judge Douglas will demonstrate somehow that this is popular sovereignty – the right of one man to make a slave of another, without any right in that other, or anyone else to object – demonstrate it as Euclid demonstrated propositions, there is no objection.'

When Douglas claimed that owners could take slaves into areas that prohibited slavery, such as the territories of Kansas and Nebraska, Lincoln drew him into a contradiction. 'He admitted at first that the slave might be lawfully taken into the Territories under the Constitution of the United States,' Lincoln said in 1859, 'and yet asserted that he might be lawfully driven out. That being the proposition, it is the absurdity I have stated.' Douglas's aim had been to cultivate support from the pro-slavery South, as well as the Northern states that wanted to halt its expansion. Lincoln had instead lured him into a logical conclusion that angered both sides. Although Douglas had won re-election to the Senate in 1858, Lincoln's Euclid-inspired traps would damage Douglas's 1860 presidential bid, while also helping his own campaign.[27]

On that slushy, snowy New York night, in a move that would be reprinted across the country, Lincoln set out his case. It was February 1860 and he opened by quoting a speech Douglas had given in Ohio the previous autumn. On the topic of slavery, Douglas had said, 'Our fathers, when they framed the Government under which we live, understood this question just as well, and even better, than we do now.'[28]

Lincoln chose to open with this quote because, he said, 'it furnishes a precise and an agreed starting point for a discussion'. Like a lawyer – or perhaps a mathematician – Lincoln began by defining the terms he'd just quoted. He carefully outlined the 'frame of government' (i.e. the US constitution), the identity of the 'fathers' (i.e. the thirty-nine men who signed the constitution), and the 'question to be understood' (i.e. whether the federal government could control slavery in its territories).

At first glance, a modern observer might conclude that the Founding Fathers overwhelmingly supported the growth of slavery. Thomas Jefferson, after all, kept people enslaved and had six daughters by one of the slaves he'd inherited from his father-in-law. But dissecting the historical actions of those thirty-nine men, Lincoln noted that once the first government was formed, the majority of them had voted to prevent the further spread of slavery. It was absurd, Lincoln said, to claim the founders didn't think slavery should be controlled, given they acted to control it. Even the constitution itself conspicuously omitted any mention of 'slave' as a concept; Lincoln argued this must have been deliberate, to avoid endorsing the idea that people could be property.[29]

Shortly after, Lincoln honed his message further in a speech in New Haven, delivered a few days before meeting Revd Gulliver on that train. 'It is easy to demonstrate that "our fathers, who framed this government under which we live", looked on slavery as wrong,' Lincoln said, 'and so framed it and everything about it as to square with the idea that it was wrong.'[30]

While the Founding Fathers had abandoned monarchy in favour of natural rights, some European monarchs tried to tighten their grip on power by embracing Enlightenment philosophy. From Spain to Austria, rulers with absolute control preached a hybrid that would become known as 'enlightened absolutism'; they would keep the power but dispense it according to Enlightenment ideals.

This contradictory notion would sometimes descend into chaos. In the case of Frederick the Great of Prussia, the trigger would be a dispute involving a trio of carp ponds. Since 1762, Christian and Rosine Arnold had owned a grain mill in the village of Pommerzig, in what is now western Poland. But in 1778, they were evicted from their land after failing to keep up the lease payments. According to the Arnolds, their business had come under threat from a nobleman who owned the land upstream. To keep his carp ponds full, the nobleman had spent the preceding eight years redirecting the water, and with it the power needed to keep the Arnolds' mill going.

In the years before their eviction, word of the Millers Arnold had spread widely, with their plight eventually reaching the attention of King Frederick. He triggered a royal commission, which found that local judges had time and again ruled against the Arnolds. Frederick became convinced that the judges had allowed an injustice to spread. Summoning them to Berlin for a scolding, Frederick explained he would be overturning the judgements. He said that courts guilty of injustice were more dangerous than thieves because the public had little hope of defending themselves. 'They are worse than the biggest scoundrels in the world, and deserve a double punishment.'[31]

When his chancellor protested at the meddling, Frederick turned on him too.[32] 'March!' he announced. 'Your post has already been given to someone else!' As well as leaving his job, the chancellor would find himself marching into prison, joining the judges and several other local officials for a year in Berlin's Spandau Citadel. Their stay turned out to be anything but isolated. Many in Berlin frowned on the king's intervention and supportive visitors would turn up regularly at the prison with drinks and supplies.

Overturning and imprisoning judges on a whim was not a particularly good demonstration of Enlightenment values, and this is perhaps why Frederick decided to follow up by finally launching his most ambitious reform. Decades earlier, he had pondered the possibility of introducing rigour into a fragmented legal system by developing a 'Prussian Code'. The idea was to have rules 'based solely upon reason and the constitutions of the provinces'.[33]

Frederick's thinking during this period was influenced by Voltaire, who, having fallen out of favour in France, increasingly visited Prussia and moved there in 1750.[34] Although their friendship would later fade, it had started intensely. 'I am sure to faint from joy,' Voltaire wrote before their first meeting in 1740. 'I believe I shall die from it,' Frederick replied.[35] Torture, which Voltaire disagreed with, would be abolished. Religious minorities would be tolerated. And then came Frederick's grandest plan.

The aim of the Prussian Code was to construct a detailed logical structure to cover any possible legal situation that could arise. The final version, which came into effect in 1794, contained over 17,000 articles of law. Frederick, who'd died eight years earlier, would never see his ambition put to the test, nor would he see it subsequently fail. Whereas Euclid opened his *Elements* with definitions of vital concepts, technical terms in the Prussian code were left undefined; to understand the new code, readers often needed to refer to the previous system of Roman Law.[36] There were also many scenarios for which the code provided no answer. Judges were not allowed to refer to the precedent of historical decisions, so instead they had to use their own interpretation. Legal uncertainty, it seemed, could not be removed through sheer brute force.[37]

Napoleon, who took power in France in 1799, had at first toyed with taking the opposite approach to the Prussians. His idea was to condense the principles of law into a few simple, fixed rules, as in mathematics.[38] The result would ideally resemble Euclid's *Elements*, with a small number of guiding principles able to address a wide range of problems. However, he abandoned the attempt after going through the existing laws with civil servants. The final Napoleonic Code contained 2,281 articles, and even this ended up deferring to several earlier legal systems.[39]

As the nineteenth century went on, others continued to search for a more scientific approach to the law. In England, legal theorist John Austin believed that tricky legal concepts could be reduced to simpler notions, which would eventually distil down into a set of fundamental axioms common to all legal systems.[40] Although Austin criticised the 'glaring deficiencies' in the French and Prussian codes, and the 'profound ignorance' of some of the authors, he argued that the codes still left the legal systems of these countries better off than before.[41]

Efforts to 'mathematicise' the law had borrowed from Euclid, but there was a catch: Euclidean logic relied on the concept of common axioms agreed upon by all. In the United States, this presented Abraham Lincoln with a dilemma. Speaking in the city of Peoria in October 1854, he had become frustrated with Stephen Douglas's view that Illinois was not founded as a free state. 'To deny these things is to deny our national axioms,' Lincoln said. Which, in his view, made argument hopeless. 'If a man will stand up and assert, and repeat, and re-assert, that two and two do not make four, I know nothing in the power of argument that can stop him.'[42]

By the end of the 1850s, slavery had drawn a dividing line through the United States. When Lincoln was unable to attend an event to mark the birthday of Jefferson, he wrote to the organisers, lamenting how swathes of his country were endorsing slavery and ignoring Jefferson's legacy. 'One would start with great confidence that he could convince any sane child that the simpler propositions of Euclid are true,' Lincoln wrote, 'but, nevertheless, he would fail, utterly, with one who should deny the definitions and axioms. The principles of Jefferson are the definitions and axioms of free society. And yet they are denied and evaded, with no small show of success.'[43]

Despite failing to win Douglas's Senate seat in 1858, two years later Lincoln ran against him for the presidency and won. It would be a decisive moment. Within months, several Southern states had split from the Union. Soon after, the nation would descend into civil war, fractured by its founding axioms. Euclid's *Elements* had guided thinkers for centuries, providing the building blocks for scientific progress and democratic development. But these foundations now had a visible flaw, and the resulting fault lines would shake science as well as politics.

2

LOGIC MAKES MATHEMATICAL MONSTERS

Around a hundred years before Euclid wrote *The Elements*, Zeno of Elea had tried unsuccessfully to topple Nearchus, the tyrant who ruled the city of Elea in southern Italy. Arrested and tortured to reveal the names of his fellow conspirators, philosopher Zeno responded by flippantly listing Nearchus's friends. The ordeal continued, but Zeno didn't budge. Instead, legend has it that he bit off his own tongue and spat it at his captor.[1]

A contemporary of Plato and Socrates, Zeno would leave a legacy of frustration. Before meeting his fate at the hands of Nearchus, he'd invented a series of logical riddles that would puzzle students and scholars for centuries. The best known of these riddles is 'Achilles and the tortoise'. It starts with the pair lining up for a foot race. To keep things fair, the tortoise gets a head start. Let's imagine the pair take their positions behind two ancient Greek starting gates, with the tortoise one mile ahead. When the gates open, Achilles quickly covers the mile gap, reaching the tortoise's starting position. In this time, though, the tortoise has moved on, say by a few feet. It doesn't take Achilles long to reach this point, but again the tortoise has lumbered on.

Zeno suggested this leads to a paradox: whenever Achilles