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The Economist

NUMBERS GUIDE

The Essentials of Business Numeracy

Sixth edition

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Introduction

"Statistical thinking will one day be as necessary a
qualification for efficient citizenship as
the ability to read and write."

H.G. Wells

THIS BOOK is about solving problems and making decisions using numerical methods. Everyone – people in business, social administrators, bankers – can do their jobs better if equipped with such tools. No special skills or prior knowledge are required. Numerical methods amount to little more than applied logic: they all reduce to step-by-step instructions and can be processed by simple computing devices. Yet numerical methods are exciting and powerful. They work magic, which is perhaps why they are shrouded in mystery. This book strips away that mystery and provides a guided tour through the statistical workshop. There are no secrets, no barriers to entry. Anyone can use these tools. Everyone should.

What are numerical methods?

Numerical methods range from the simple (how to calculate percentages and interest) to the relatively complex (how to evaluate competing investment opportunities); from the concrete (how to find the shortest route for deliveries) to the vague (how to deal with possible levels of sales or market share). The link is quantitative analysis, a scientific approach.

This does not mean that qualitative factors (intangibles such as personal opinion, hunch, technological change and environmental awareness) should be ignored. On the contrary, they must be brought into the decision process, but in a clear, unemotional way. Thus, a major part of this book is devoted to dealing with risk. After all, this forms a major part of the business environment. Quantifying risk and incorporating it into the decision-making process is essential for successful business.

In bringing together quantitative techniques, the book borrows heavily from mathematics and statistics and also from other fields, such as accounting and economics.

A brief summary

We all perform better when we understand why we are doing something. For this reason, this book always attempts to explain why as well as how methods work. Proofs are not given in the rigorous mathematical sense, but the techniques are explained in such a way that the reader should be able to gain at least an intuitive understanding of why they work. This should also aid students who use this book as an introduction to heavier statistical or mathematical works.

The techniques are illustrated with business examples where possible but sometimes abstract illustrations are preferred. This is particularly true of probability, which is useful for assessing business risk but easier to understand through gamblers' playing cards and coins.

Examples use many different currencies and both metric and imperial measurements. The SI standards for measurement (see SI units in the A–Z) are excellent, but they are generally ignored here in favour of notation and units which may be more familiar to a wider audience.

This book works from the general to the particular.

Chapter 1 lays the groundwork by running over some key concepts. Items of particular interest include proportions and percentages (which appear in many problems) and probability (which forms a basis for assessing risk).

Chapter 2 examines ways of dealing with problems and decisions involving money, as many or most do. Interest, inflation and exchange rates are all covered. Note that the proportions met in the previous chapter are used as a basis for calculating interest and evaluating investment projects.

Chapter 3 looks at summary measures (such as averages) which are important tools for interpretation and analysis. In particular, they unlock what is called the normal distribution, which is invaluable for modelling risk.

Chapter 4 reviews the way data are ordered and interpreted using charts and tables. A series of illustrations draws attention to the benefits and shortfalls of various types of presentation.

Chapter 5 examines the vast topic of forecasting. Few jobs can be done successfully without peering into the future. The objective is to pull together a view of the future in order to enhance the inputs to decision-making.

Chapter 6 marks a turning point. It starts by considering the way that sampling saves time and money when collecting the inputs to decisions. This is a continuation of the theme in the previous chapters. However,

the chapter then goes on to look at ways of reaching the best decision from sample data. The techniques are important for better decision-making in general.
Chapter 7 expands on the decision theme. It combines judgment with the rigour of numerical methods for better decisions in those cases which involve uncertainty and risk.
Chapter 8 looks at some rather exciting applications of techniques already discussed. It covers:

- game strategy (for decision-making in competitive situations);
- queueing (for dealing with a wide range of business problems, only one of which involves customers waiting in line);
- stock control (critical for minimising costs);
- Markov chains (for handling situations where events in the future are directly affected by preceding events);
- project management (with particular attention to risk); and
- simulation (for trying out business ideas without risking humiliation or loss).

Chapter 9 reviews powerful methods for reaching the best possible decision when risk is not a key factor.
An A–Z section concludes the book. It gives key definitions. It covers a few terms which do not have a place in the main body of the book. And it provides some useful reference material, such as conversion factors and formulae for calculating areas and volumes.

Additional information is available on this book's website at www.NumbersGuide.com.

How to use this book

There are four main approaches to using this book:

1. If you want to know the meaning of a mathematical or statistical term, consult the A–Z. If you want further information, turn to the cross-reference in the A–Z entry, shown in small capital letters, and read more.
2. If you want to know about a particular numerical method, turn to the appropriate chapter and read all about it.
3. If you have a business problem that needs solving, use the A–Z, the contents page, or this chapter for guidance on the methods available, then delve deeper.

4 If you are familiar with what to do but have forgotten the detail, then formulae and other reference material are highlighted throughout the book.

Calculators and computers

There can be few people who are not familiar with electronic calculators. If you are selecting a new calculator for use on your computer or hand-held device, choose one with basic operations (+ − × and ÷) and at least one memory, together with the following:

- Exponents and roots (probably summoned by keys marked x^y and $x^{1/y}$): essential for dealing with growth rates, compound interest and inflation.
- Factorials (look for a key such as x!): useful for calculating permutations and combinations.
- Logarithms (log and 10^x or ln and e^x): less important but sometimes useful.
- Trigonometric functions (sin, cos and tan): again, not essential, but handy for some calculations (see Triangles and trigonometry in the A–Z).
- Base/radix conversion/operations: useful for exploring bases, and for doing boffinish stuff with computers.
- Constants π and e: occasionally useful.
- Net present value and internal rate of return (NPV and IRR). These are found only on financial calculators. They assist investment evaluation, but you will probably prefer to use a spreadsheet.

Note that the calculator in Microsoft Windows has most of the above functions when used in scientific mode.

Computer and maybe tablet users will find themselves turning to a spreadsheet program to try out many of the techniques in this book. Spreadsheets take the tedium out of many operations and are more or less essential for some activities such as simulation.

For the non-initiated, a spreadsheet is like a huge sheet of blank paper divided up into little boxes (known as cells). You can key text, numbers or instructions into any of the cells. If you enter ten different values in a sequence of ten cells, you can then enter an instruction in the eleventh, perhaps telling the computer to add or multiply the ten values together. One powerful feature is that you can copy things from one cell to another almost effortlessly. Tedious, repetitive tasks become simple. Another handy feature is the large selection of instructions (or functions) which enable you to do much more complex things than you would with a calculator. Lastly, spreadsheets also produce charts which are handy for interpretation and review.

The market leader in spreadsheet programs is Microsoft Excel (packaged with Microsoft Office). It is on the majority of corporate desktops. Good alternatives to MS Office, which happen to be free and include spreadsheets broadly compatible with Excel, include OpenOffice and Google Docs.

Conclusion

There are so many numerical methods and potential business problems that it is impossible to cross-reference them all. Use this book to locate techniques applicable to your problems and take the following steps:

- Define the problem clearly.
- Identify the appropriate technique.
- Collect the necessary data.
- Develop a solution.
- Analyse the results.
- Start again if necessary or implement the results.

The development of sophisticated computer packages has made it easy for anyone to run regressions, to identify relationships or to make forecasts. But averages and trends often conceal more than they reveal. Never accept results out of hand. Always question whether your analysis may have led you to a faulty solution. For example, you may be correct in noting a link between national alcohol consumption and business failures; but is one directly related to the other, or are they both linked to some unidentified third factor?

1 Key concepts

"Round numbers are always false."

Samuel Johnson

Summary

Handling numbers is not difficult. However, it is important to be clear about the basics. Get them right and everything else slots neatly into place.

People tend to be comfortable with percentages, but it is easy to perform many calculations using proportions. The two are used interchangeably throughout this book. When a result is obtained as a proportion, such as 6 ÷ 100 = 0.06, this is often referred to as 6%. Sums become much easier when you can convert between percentages and proportions by eye: just shift the decimal point two places along the line (adding zeros if necessary).

Proportions simplify problems involving growth, reflecting perhaps changes in demand, interest rates or inflation. Compounding by multiplying by one plus a proportion several times (raising to a power) is the key. For example, compound growth of 6% per year over two years increases a sum of money by a factor of 1.06 × 1.06 = 1.1236. So $100 growing at 6% per year for two years increases to $100 × 1.1236 = $112.36.

Proportions are also used in probability, which is worth looking at for its help in assessing risks.

Lastly, index numbers are introduced in this chapter.

Ways of looking at data

It is useful to be aware of different ways of categorising information. This is relevant for three main reasons.

1 *Time series and cross-sectional data*

Certain problems are found with some types of data only. For example, it is not too hard to see that you would look for seasonal and cyclical trends in time series but not in cross-sectional data.

Time series record developments over time; for example, monthly ice cream output, or a ten-year run of the finance director's annual salary.

Cross-sectional data are snapshots that capture a situation at a moment in time, such as the value of sales at various branches on one day.

2 ***Scales of measurement***

Some techniques are used with one type of data only. A few of the sampling methods in Chapter 7 are used only with data which are measured on an interval or ratio scale. Other sampling methods apply to nominal or ordinal scale information only.

Nominal or categorical data identify classifications only. No particular quantities are implied. Examples include sex (male/female), departments (international/marketing/personnel) and sales regions (area number 1, 2, 3, 4).

Ordinal or ranked data. Categories can be sorted into a meaningful order, but differences between ranks are not necessarily equal. What do you think of this politician (awful, satisfactory, wonderful)? What grade of wheat is this (A1, A2, B1...)?

Interval scale data. Measurable differences are identified, but the zero point is arbitrary. Is 20° Celsius twice as hot as 10°C? Convert to Fahrenheit to see that it is not. The equivalents are 68°F and 50°F. Temperature is measured on an interval scale with arbitrary zero points (0°C and 32°F).

Ratio scale data. There is a true zero and measurements can be compared as ratios. If three frogs weigh 250gm, 500gm and 1,000gm, it is clear that Mr Frog is twice as heavy as Mrs Frog, and four times the weight of the baby.

3 ***Continuity***

Some results are presented in one type of data only. You would not want to use a technique which tells you to send 0.4 of a salesman on an assignment, when there is an alternative technique which deals in whole numbers.

Discrete values are counted in whole numbers (integers): the number of frogs in a pond, the number of packets of Fat Cat Treats sold each week.

Continuous variables do not increase in steps. Measurements such as heights and weights are continuous. They can only be estimated: the temperature is 25°C; this frog weighs 500gm. The accuracy of such estimates depends on the precision of the measuring instrument. More accurate scales might show the weight of the frog at 501 or 500.5 or 500.0005 gm, etc.

Fractions, percentages and proportions

Fractions

Fractions are not complicated. Most monetary systems are based on 100 subdivisions: 100 cents to the dollar or euro, or 100 centimes to the Swiss franc. Amounts less than one big unit are fractions; 50 cents is half, or 0.50, or 50% of one euro. Common (vulgar) fractions (½), decimal fractions (0.50), proportions (0.50) and percentages (50%) are all the same thing with different names. Convert any common fraction to a decimal fraction by dividing the lower number (denominator) into the upper number (numerator). For example, ¾ = 3 ÷ 4 = 0.75. The result is also known as a proportion. Multiply it by 100 to convert it into a percentage. Recognition of these simple relationships is vital for easy handling of numerical problems.

Decimal places. The digits to the right of a decimal point are known as decimal places. 1.11 has two decimal places, 1.111 has three, 1.1111 has four, and so on.

Reading decimal fractions. Reading $10.45m as ten-point-forty-five million dollars will upset the company statistician. Decimal fractions are read out figure-by-figure: ten-point-four-five in this example. Forty-five implies four tens and five units, which is how it is to the left of

Percentage points and basis points

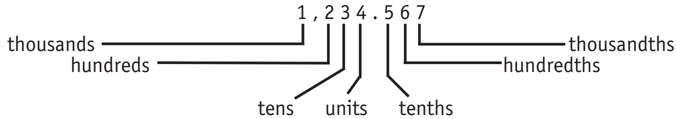
Percentages and percentage changes are sometimes confused. If an interest rate or inflation rate increases from 10% to 12%, it has risen by two units, or two percentage points. But the percentage increase is 20% (= 2 ÷ 10 × 100). Take care to distinguish between the two.

Basis points. Financiers attempt to profit from very small changes in interest or exchange rates. For this reason, one unit, say 1% (ie, one percentage point) is often divided into 100 basis points:

1	basis point	=	0.01 percentage point
10	basis points	=	0.10 percentage point
25	basis points	=	0.25 percentage point
100	basis points	=	1.00 percentage point

Number values

1.1



Common fraction	Decimal equivalent
5/10	0.5
56/100	0.56
567/1000	0.567

Percentage increases and decreases

A percentage increase followed by the same percentage decrease does not leave you back where you started. It leaves you worse off. Do not accept a 50% increase in salary for six months, to be followed by a 50% cut.

- $1,000 increased by 50% is $1,500.
- 50% of $1,500 is $750.

A frequent business problem is finding what a number was before it was increased by a given percentage. Simply divide by (1 + i), where i is the percentage increase expressed as a proportion. For example:

- if an invoice is for €575 including 15% VAT (value added tax, a sales tax) the tax-exclusive amount is €575 ÷ 1.15 = €500.

Fractions. If anything is increased by an amount x/y, the increment is $x/(x + y)$ of the new total:

- if €100 is increased by $1/2$, the increment of €50 is $1/(1 + 2) = 1/3$ of the new total of €150;
- ¥100 increased by $3/4$ is ¥175; the ¥75 increment is $3/(3 + 4) = 3/7$ of the new ¥175 total.

the decimal point. To the right, the fractional amounts shrink further to tenths, hundredths, and so on. (See Figure 1.1.)

Think of two fractions. It is interesting to reflect that fractions go on for ever. Think of one fractional amount; there is always a smaller one. Think of two fractions; no matter how close together they are, there is

How big is a billion?

As individuals we tend to deal with relatively small amounts of cash. As corporate people, we think in units of perhaps one million at a time. In government, money seemingly comes in units of one billion only.

Scale. The final column below, showing that a billion seconds is about 32 years, gives some idea of scale. The fact that Neanderthal man faded away a mere one trillion seconds ago is worth a thought.

Quantity	*Zeros*	*Scientific*	*In numbers*	*In seconds*
Thousand	3	1×10^3	1,000	17 minutes
Million	6	1×10^6	1,000,000	11 ½ days
Billion	9	1×10^9	1,000,000,000	32 years
Trillion	12	1×10^{12}	1,000,000,000,000	32 thousand years

British billions. The number of zeros shown are those in common use. The British billion (with 12 rather than 9 zeros) is falling out of use. It is not used in this book.

Scientific notation. Scientific notation can be used to save time writing out large and small numbers. Just shift the decimal point along by the number of places indicated by the exponent (the little number in the air). For example:

- 1.25×10^6 is a shorthand way of writing 1,250,000;
- 1.25×10^{-6} is the same as 0.00000125.

Some calculators display very large or very small answers this way. Test by keying 1 ÷ 501. The calculator's display might show 1.996 −03, which means 1.996×10^{-3} or 0.001996. You can sometimes convert such displays into meaningful numbers by adding 1. Many calculators showing 1.996 −03 would respond to you keying + 1 by showing 1.001996. This helps identify the correct location for the decimal point.

always another one to go in between. This brings us to the need for rounding.

Rounding

An amount such as $99.99 is quoted to two decimal places when selling, but usually rounded to $100 in the buyer's mind. The Japanese have stopped counting their sen. Otherwise they would need wider calculators. A few countries are perverse enough to have currencies with three places of decimal: 1,000 fils = 1 dinar. But 1 fil coins are generally no longer in use and values such as 1.503 are rounded off to 1.505. How do you round 1.225 if there are no 5 fil coins? It depends whether you are buying or selling.

Generally, aim for consistency when rounding. Most calculators and spreadsheets achieve this by adopting the 4/5 principle. Values ending in 4 or less are rounded down (1.24 becomes 1.2), amounts ending in 5 or more are rounded up (1.25 becomes 1.3). Occasionally this causes problems.

Two times two equals four. Wrong: the answer could be anywhere between two and six when dealing with rounded numbers.

- 1.5 and 2.4 both round to 2 (using the 4/5 rule)
- 1.5 multiplied by 1.5 is 2.25, which rounds to 2
- 2.4 multiplied by 2.4 is 5.76, which rounds to 6

Also note that 1.45 rounds to 1.5, which rounds a second time to 2, despite the original being nearer to 1.

The moral is that you should take care with rounding. Do it after multiplying or dividing. When starting with rounded numbers, never quote the answer to more significant figures (see below) than the least precise original value.

Significant figures

Significant figures convey precision. Take the report that certain American manufacturers produced 6,193,164 refrigerators in a particular year. For some purposes, it may be important to know that exact number. Often, though, 6.2m, or even 6m, conveys the message with enough precision and a good deal more clarity. The first value in this paragraph is quoted to seven significant figures. The same amount to two significant figures is 6.2m (or 6,200,000). Indeed, had the first amount been estimated from refrigerator-makers' turnover and the average sale price of a refrigerator, seven-figure approximation would be spurious accuracy, of which economists are frequently guilty.

Significant figures and decimal places in use. Three or four significant figures with up to two decimal places are usually adequate for discussion purposes, particularly with woolly economic data. (This is sometimes called three or four effective figures.) Avoid decimals where possible, but do not neglect precision when it is required. Bankers would cease to make a profit if they did not use all the decimal places on their calculators when converting exchange rates.

Percentages and proportions

Percentages and proportions are familiar through money: 45 cents is 45% of 100 cents, or, proportionately, 0.45 of one dollar. Proportions are expressed relative to one, percentages in relation to 100. Put another way, a percentage is a proportion multiplied by 100. This is a handy thing to know when using a calculator.

Suppose a widget which cost $200 last year now retails for $220. Proportionately, the current cost is 1.1 times the old price ($220 \div 200 = 1.1$). As a percentage, it is 110% of the original ($1.1 \times 100 = 110$).

In common jargon, the new price is 10% higher. The percentage increase (the 10% figure) can be found in any one of several ways. The most painless is usually to calculate the proportion ($220 \div 200 = 1.1$); subtract 1 from the answer ($1.10 - 1 = 0.10$); and multiply by 100 ($0.10 \times 100 = 10$). Try using a calculator for the division and doing the rest by eye; it's fast.

Proportions and growth. The relationship between proportions and percentages is astoundingly useful for compounding.

The finance director has received annual 10% pay rises for the last ten years. By how much has her salary increased? Not 100%, but nearly 160%. Think of the proportionate increase. Each year, she earned 1.1 times the amount in the year before. In year one she received the base amount (1.0) times $1.1 = 1.1$. In year two, total growth was $1.1 \times 1.1 = 1.21$. In year three, $1.21 \times 1.1 = 1.331$, and so on up to $2.358 \times 1.1 = 2.594$ in the tenth year. Take away 1 and multiply by 100 to reveal the 159.4 percentage increase over the whole period.

Powers. The short cut when the growth rate is always the same, is to recognise that the calculation involves multiplying the proportion by itself a number of times. In the previous example, 1.1 was multiplied by itself 10 times. In math-speak, this is called raising 1.1 to the power of 10 and is written 1.1^{10}.

The same trick can be used to "annualise" monthly or quarterly rates of growth. For example, a monthly rise in prices of 2.0% is equivalent to an annual rate of inflation of 26.8%, not 24%. The statistical section in the back of *The Economist* each week shows for over 40 countries the annualised percentage changes in output in the latest quarter compared with the previous quarter. If America's GDP is 1.7% higher during the January–March quarter than during the October–December quarter, then this is equivalent to an annual rate of increase of 7% (1.017 × 1.017 × 1.017 × 1.017).

Using a calculator. Good calculators have a key marked something like x^y, which means x (any number) raised to the power of y (any other number). Key 1.1 x^y 10 = and the answer 2.5937... pops up on the display. It is that easy. To go back in the other direction, use the $x^{1/y}$ key. So 2.5937 $x^{1/y}$ 10 = gives the answer 1.1. This tells you that the number that has to be multiplied by itself 10 times to give 2.5937 is 1.1. (See also Growth rates and exponents box in Chapter 2.)

Table 1.1 **Mr and Mrs Average's shopping basket**

	A	B	C
	€	*Jan 2012 = 100*	*Jan 2014 = 100*
January 2012	1,568.34	100.0	86.6
January 2013	1,646.76	105.0	90.9
January 2014	1,811.43	115.5	100.0

Each number in column B = number in column A divided by (1,568.34 ÷ 100).
Each number in column C = number in column B divided by (1,811.43 ÷ 100).

Table 1.2 **A base-weighted index of living costs**

	A	B	C
	Food	*Other*	*Total*
Weights:	0.20	0.80	1.00
January 2012	100.0	100.0	100.0
January 2013	103.0	105.5	105.0
January 2014	108.0	117.4	115.5

Each monthly value in column C = (column A × 0.20) + (column B × 0.80).
Eg, for January 2014 (108.0 × 0.20) + (117.4 × 0.80) = 115.5.

Table 1.3 **A current-weighted index of living costs**

	A *Food* *index*	*B* *Food* *weight*	*C* *Other* *index*	*D* *Other* *weight*	*E* *Total*
January 2012	100.0	0.80	100.0	0.20	100.0
January 2013	103.0	0.70	105.5	0.30	103.8
January 2014	108.0	0.60	117.4	0.40	111.8

Each value in column E is equal to (number in column A × weight in column B) + (number in column C × weight in column D).
Eg, for January 2014 (108.0 × 0.60) + (117.4 × 0.40) = 111.8

Index numbers

There comes a time when money is not enough, or too much, depending on how you look at it. For example, the consumer prices index (also known as the cost of living or retail prices index) attempts to measure inflation as experienced by Mr and Mrs Average. The concept is straightforward: value all the items in the Average household's monthly shopping basket; do the same at some later date; and see how the overall cost has changed. However, the monetary totals, say €1,568.34 and €1,646.76, are not easy to handle and they distract from the task in hand. A solution is to convert them into index numbers. Call the base value 100. Then calculate subsequent values based on the percentage change from the initial amount. The second shopping basket cost 5% more, so the second index value is 105. A further 10% rise would take the index to 115.5.

To convert any series of numbers to an index:

- choose a base value (eg, €1,568.34 in the example here);
- divide it by 100, which will preserve the correct number of decimal places; then
- divide every reading by this amount.

Table 1.1 shows how this is done in practice.

Rebasing. To rebase an index so that some new period equals 100, simply divide every number by the value of the new base (Table 1.1).

Composite indices and weighting. Two or more sub-indices are often combined to form one composite index. Instead of one cost of living index for the Averages, there might be two: showing expenditure on food, and all other spending. How should they be combined?

The index number "convergence illusion" 1.2

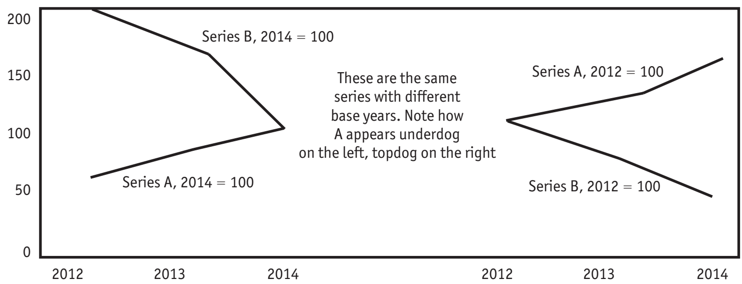


Table 1.4 **Index comparisons**

	GDP per head $	*Index US = 100*	*Index UK = 100*	*Index Germany = 100*
Monaco	171,465	356.4	439.9	389.5
Luxembourg	114,232	237.4	293.1	259.5
Norway	98,081	203.9	251.7	222.8
Qatar	92,501	192.3	237.3	210.1
Switzerland	83,326	173.2	213.8	189.3
Macao SAR, China	65,550	136.2	168.2	148.9
Kuwait	62,664	130.2	160.8	142.3
Australia	61,789	128.4	158.5	140.4
Denmark	59,889	124.5	153.7	136.0
Sweden	57,114	118.7	146.5	129.7
Canada	50,344	104.6	129.2	114.4
Netherlands	50,085	104.1	128.5	113.8
Austria	49,581	103.1	127.2	112.6
Finland	48,812	101.5	125.2	110.9
United States	48,112	100.0	123.4	109.3
Ireland	47,478	98.7	121.8	107.9
Belgium	46,608	96.9	119.6	105.9
Singapore	46,241	96.1	118.6	105.0
Japan	45,903	95.4	117.8	104.3
United Arab Emirates	45,653	94.9	117.1	103.7
Germany	44,021	91.5	112.9	100.0
Iceland	43,967	91.4	112.8	99.9
France	42,379	88.1	108.7	96.3
United Kingdom	38,974	81.0	100.0	88.5

Base weighting. The most straightforward way of combining indices is to calculate a weighted average. If 20% of the budget goes on food and

80% on other items, the sums look like those in Table 1.2. Note that the weights sum to one (they are actually proportions, not percentages); this simplifies the arithmetic.

Since this combined index was calculated using weights assigned at the start, it is known as a base-weighted index. Statisticians in the know sometimes like to call it a Laspeyres index, after the German economist who developed the first one.

Current weighting. The problem with weighted averages is that the weights sometimes need revision. With the consumer prices index, spending habits change because of variations in relative cost, quality, availability and so on. Indeed, UK statisticians came under fire as early as 1947 for producing an index of retail prices using outdated weights from a 1938 survey of family expenditure habits.

One way to proceed is to calculate a new set of current weights at regular intervals, and use them to draw up a long-term index. Table 1.3 shows one way of doing this.

This current-weighted index is occasionally called a Paasche index, again after its founder.

Imperfections and variations on weighting. Neither a base-weighted nor a current-weighted index is perfect. The base-weighted one is simple to calculate, but it exaggerates changes over time. Conversely, a current-weighted index is more complex to produce and it understates long-term changes. Neither Laspeyres nor Paasche got it quite right, and others have also tried and failed with ever more complicated formulae. Other methods to be aware of are Edgeworth's (an average of base and current weights), and Fisher's (a geometric average combining Laspeyres and Paasche indices).

Mathematically, there is no ideal method for weighting indices. Indeed, indices are often constructed using weights relating to some period other than the base or current period, or perhaps an average of several periods. Usually a new set of weights is introduced at regular intervals, maybe every five years or so.

Convergence. Watch for illusory convergence on the base. Two or more series will always meet at the base period because that is where they both equal 100 (see Figure 1.2). This can be highly misleading. Whenever you come across indices on a graph, the first thing you should do is check where the base is located.

Cross-sectional data. Index numbers are used not only for time series but also for snapshots. For example, when comparing salaries or other indicators in several countries, commentators often base their figures on

Summation and factorials

Summation. The Greek uppercase S, sigma or Σ, is used to mean nothing more scary than take the sum of. So "Σ profits" indicates "add up all the separate profits figures". Sigma is sometimes scattered with other little symbols to show how many of what to take the sum of. (It is used here only when this information is self-evident.) For example, to find the average salary of these four staff take the sum of their salaries and divide by four could be written: Σ salaries ÷ 4.
Factorials. A fun operator is the factorial identified by an exclamation mark ! where 5! (read five factorial) means 5 × 4 × 3 × 2 × 1. This is useful shorthand for counting problems.

their home country value as 100. This makes it easy to rank and compare the data, as Table 1.4 shows.

Notation

When you jot your shopping list or strategic plan on the back of an envelope you use shorthand. For example, "acq WW" might mean "acquire (or take over) World of Widgets". Sometimes you borrow from the world of numbers. Symbols such as + for add and = for equals need no explanation in the English-speaking world.

So why do we all freeze solid when we see mathematical shorthand? Not because it is hard or conceptually difficult, but because it is unfamiliar. This is odd. Mathematicians have been developing their science for a few thousand years. They have had plenty of time to develop and promulgate their abbreviations. Some are remarkably useful, making it simple to define problems concisely, after which the answer is often self-evident. For this reason, this book does not fight shy of symbols, but they are used only where they aid clarity.

The basic operators + − × ÷ are very familar. Spreadsheet and other computer users will note that to remedy a keyboard famine ÷ is replaced by / and × is replaced by *. Mathematicians also sometimes use / or write one number over another to indicate division, and omit the multiplication sign. Thus if a = 6 and b = 3:

$$a \div b = a/b = {}^{a}\!/\!_{b} = \frac{a}{b} = 2 \text{ and}$$
$$a \times b = ab = a\,.\,b = 18$$

Brackets. When the order of operation is important, it is highlighted with brackets. Perform operations in brackets first. For example, $4 \times (2 + 3) = 20$ is different from $(4 \times 2) + 3 = 11$. Sometimes more than one set of brackets is necessary, such as in $[(4 \times 2) + 3] \times 6 = 66$. When entering complex formulas in spreadsheet cells, always use brackets to ensure that the calculations are performed as intended.
Powers. When dealing with growth rates (compound interest, inflation, profits), it is frequently necessary to multiply a number by itself many times. Writing this out in full becomes laborious. To indicate, for example, $2 \times 2 \times 2 \times 2$, write 2^4 which is read "two raised to the power of four". The little number in the air is called an exponent (Latin for out-placed).
Roots. Just as the opposite of multiplication is division, so the opposite of raising to powers is taking roots. $\sqrt[4]{625}$ is an easy way to write "take the fourth root of 625" – or in this case, what number multiplied by itself four times equals 625? (Answer 5, since $= 5 \times 5 \times 5 \times 5 = 625$) The second root, the square root, is generally written without the 2 (eg, $\sqrt[2]{9} = \sqrt{9} = 3$). Just to confuse matters, a convenient alternative way of writing "take a root" is to use one over the exponent. For example, $16^{1/4} = \sqrt[4]{16} = 2$.
Equalities and inequalities. The equals sign = (or equality) needs no explanation. Its friends, the inequalities, are also useful for business problems. Instead of writing "profits must be equal to or greater than ¥5m", scribble "profits ≥ ¥5m". The four inequalities are:

- greater than or equal to $\geq$
- less than or equal to $\leq$
- greater than $>$
- less than $<$

They are easy to remember since they open up towards greatness. A peasant $<$ a prince (perhaps). Along the same lines not equal $\neq$ and approximately equal $\approx$ are handy.

Symbols

Letters such as a, b, x, y and n sometimes take the place of constants or variables – things which can take constant or various values for the purpose of a piece of analysis.

For example, a company trading in xylene, yercum and zibeline (call these x, y and z), which it sells for €2, €3 and €4 a unit, would calculate sales revenue (call this w) as:

$$w = (2 \times x) + (3 \times y) + (4 \times z)$$
$$\text{or } w = 2x + 3y + 4z$$

When sales figures are known at the end of the month, the number of units sold of xylene, yercum and zibeline can be put in place of x, y and z in the equation so that sales revenue w can be found by simple arithmetic. If sales prices are as yet undetermined, the amounts shown above as €2, €3 and €4 could be replaced by a, b and c so that the relationship between sales and revenue could still be written down:

$$w = (a \times x) + (b \times y) + (c \times z)$$
$$\text{or } w = ax + by + cz$$

See below for ways of solving such equations when only some of the letters can be replaced by numerical values.

When there is a large number of variables or constants, there is always a danger of running out of stand-in letters. Alternative ways of rewriting the above equation are:

$$w = (a \times x_1) + (b \times x_2) + (c \times x_3)$$
$$\text{or even } x_0 = (a_1 \times x_1) + (a_2 \times x_2) + (a_3 \times x_3)$$

The little numbers below the line are called subscripts, where x_1 = xylene, x_2 = yercum, and so on.

General practice. There are no hard and fast rules. Lowercase letters near the beginning of the alphabet (a, b, c) are generally used for constants, those near the end (x, y, z) for variables. Frequently, y is reserved for the major unknown which appears on its own on the left-hand side of an equation, as in $y = a + (b \times x)$. The letter n is often reserved for the total number of observations, as in "we will examine profits over the last n months" where n might be set at 6, 12 or 24.

Romans and Greeks. When the Roman alphabet becomes limiting, Greek letters are called into play. For example, in statistics, Roman letters are used for sample data (p = proportion from a sample). Greek equivalents indicate population data (π indicates a proportion from a population; see Proportions in Chapter 6).

Circles and pi. To add to the potential confusion the Greek lower-case p (π, pi) is also used as a constant. By a quirk of nature, the distance around the edge of a circle with a diameter of 1 foot is 3.14 feet. This measurement is important enough to have a name of its own. It is labelled π, or pi. That is, $\pi = 3.14$. Interestingly, pi cannot be calculated exactly. It is 3.1415927 to eight significant figures. It goes on forever and is known as an irrational number.

Solving equations

Any relationship involving an equals sign = is an equation. Two examples of equations are $3 + 9x = 14$ and $(3 \times x) + (4 \times y) = z$. The following three steps will solve any everyday equation. They may be used in any order and as often as necessary:

1 **Add or multiply.** The equals sign = is a balancing point. If you do something to one side of the equation you must do the same to the other.

Addition and subtraction

With this equation subtract 14 from both sides to isolate y:

$$y + 14 = x$$
$$(y + 14) - 14 = x - 14$$
$$y = x - 14$$

Multiplication and division

With this equation divide both sides by 2 to isolate y:

$$y \times 2 = x$$
$$(y \times 2) \div 2 = x \div 2$$
$$y = x \div 2$$

2 **Remove awkward brackets.**

$$2 \times (6 + 8) = (2 \times 6) + (2 \times 8) = 12 + 16 = 28$$
$$2 \times (x + y) = (2 \times x) + (2 \times y)$$

3 **Dispose of awkward subtraction or division.**

Subtraction is negative addition (eg, $6 - 4 = 6 + -4 = 2$).

Note that a plus and a minus is a minus (eg, $6 + -2 = 4$), while two minus signs make a plus (eg, $6 - -2 = 8$).

Division by x is multiplication by the reciprocal $1/x$ (eg, $6 \div 3 = 6 \times 1/3 = 2$).

Note that $3 \times 1/3 = 3 \div 3 = 1$.

For example, suppose that in the following relationship, y is to be isolated:

$$w/6 = (2 \times y) + (12 \times x) - 3$$

Subtract (12 × x) from both sides:

$$w/6 - (12 \times x) = (2 \times y) + (12 \times x) - (12 \times x) - 3$$
$$w/6 - (12 \times x) = (2 \times y) - 3$$

Add 3 to both sides:

$$w/6 - (12 \times x) + 3 = (2 \times y) - 3 + 3$$
$$w/6 - (12 \times x) + 3 = (2 \times y)$$

Multiply both sides by ½:

$$1/2 \times [w/6 - (12 \times x) + 3] = 1/2 \times (2 \times y)$$
$$\text{or } w/12 - (6 \times x) + 3/2 = y$$
$$\text{ie, } y = w/12 - (6 \times x) + 3/2$$

Probability

Rationalising uncertainty

Uncertainty dominates business (and other) decisions. Yet there is a certainty about uncertainty that makes it predictable; it is possible to harness probability to improve decision-making.

For example, right now, someone might be stealing your car or burgling your home or office. If you are not there, you cannot be certain, nor can your insurance company. It does not know what is happening to your neighbour's property either, or to anyone else's in particular. What the insurance company does know from experience is that a given number (1, 10, 100 ...) of its clients will suffer a loss today.

Take another example: toss a coin. Will it land heads or tails? Experience or intuition suggests that there is a 50:50 chance of either. You cannot predict the outcome with certainty. You will be 100% right or 100% wrong. But the more times you toss it, the better your chance of predicting in advance the proportion of heads. Your percentage error tends to reduce as the number of tosses increases. If you guess at 500 heads before a marathon 1,000-toss session, you might be only a fraction of a per cent out.

The law of large numbers. In the long run, the proportion (relative frequency) of heads will be 0.5. In math-speak, probability is defined as the limit of relative frequency. The limit is reached as the number of repetitions approaches infinity, by which time the proportion of heads should be fairly and squarely at 0.500. The tricky part is that you can never quite get to infinity. Think of a number – you can always add one more. It is similar to the "think of two fractions" problem mentioned earlier. Mathematicians are forced to say it is probable that the limit of relative frequency will be reached at infinity. This is called the law of large numbers, one of the few theorems with a sensible name. It seems to involve circular reasoning (probability probably works at infinity). But there is a more rigorous approach through a set of laws (axioms) which keeps academics happy.

Applications. In many cases, probability is helpful for itself. Quantifying the likelihood of some event is useful in the decision-making process. But probability also forms the basis for many interesting decision-making techniques discussed in the following chapters. The ground rules are considered below.

Estimating probabilities

Measuring. Everyday gambling language (10 to 1 odds on a horse, a 40% chance of rain) is standardised in probability-speak.

Probability is expressed on a sliding scale from 0 to 1. If there is no chance of an event happening, it has a probability of zero. If it must occur, it has a probability of 1 (this is important). An unbiased coin can land on a flat surface in one of only two ways. There is a 50% chance of either. Thus, there is a 0.5 probability of a head, and a 0.5 probability of a tail. If four workers are each equally likely to be selected for promotion, there is a 1 in 4, or 25%, chance that any one will be selected. They each have a 0.25 probability of rising further towards their level of incompetence. Probabilities are logic expressed proportionately.

Certainty = 1. Why highlight the importance of an unavoidable event having a probability of one? Look at the coin example. There is a 0.5 probability of a head and a 0.5 probability of a tail. One of these two events must happen (excluding the chances of the coin staying in the air or coming to rest on its edge), so the probabilities must add up to one.

If the probability of something happening is known, then by definition the probability of it not happening is also known. If the meteorologist's new computer says there is a 0.6 probability of rain tomorrow, then there is a 0.4 probability that it will not rain. (Of course, the meteorologist's computer might be wrong.)

Assigning probabilities using logic. When the range of possible outcomes can be foreseen, assigning a probability to an event is a matter of

Probability rules

The probability of any event P(A) is a number between 0 and 1, where 1 = certainty.

GENERAL RULE	EXAMPLE
The probability of event A is the number of outcomes where A happens n_A divided by the total number of possible outcomes n $P(A) = n_A \div n$.	P(red ace), $P(A) = 2/52 = 0.038$[a] P(any heart), $P(B) = 13/52 = 0.25$ P(king of clubs), $P(C) = 1/52 = 0.019$
The probability of an event not occurring is equal to the one minus the probability of it happening $P(\text{not } A) = 1 - P(A)$	Probability of drawing at random any card other than a red ace = $P(\text{not } A) = 1 - 2/52 = 50/52 = 0.962$
PROBABILITY OF COMPOSITE EVENTS	
One outcome given another	
The probability of event A given that event B is known to have occurred = number of outcomes where A happens when A is selected from B (n_A) divided by number of possible outcomes B (n_B) $P(A\|B) = n_A \div n_B$	Probability that the card is a red ace given that you know a heart was drawn = $P(A\|B) = 1/13 = 0.077$
One outcome or the other [b]	
Independent (mutually exclusive) events $P(A \text{ or } B) = P(A) + P(B)$	Probability of drawing a red ace or any club = $P(A \text{ or } B) = 2/52 + 13/52 = 15/52$
Dependent (non-exclusive) events $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$	Probability of drawing the king of clubs or any club = $P(A \text{ or } B) = 1/52 + 13/52 - 1/52 = 13/52$
Both outcomes [b]	
Independent (mutually exclusive) events $P(A \text{ and } B) = P(A) \times P(B)$	Probability of drawing a red ace, returning it to the pack and then drawing any heart = $P(A \text{ and } B) = 2/52 \times 13/52 = 1/104$
Dependent (non-exclusive) events $P(A \text{ and } B) = P(A) \times P(B\|A)$ or $P(A \text{ and } B) = P(B) \times P(A\|B)$	Probability of drawing one card which is both a red ace and a heart = $P(A \text{ and } B) = 2/52 \times 1/2 = 1/52$ or $P(A \text{ and } B) = 13/52 \times 1/13 = 1/52$

a There are 52 cards in a pack, split into four suits of 13 cards each. Hearts and diamonds are red, while the other two suits, clubs and spades, are black.
b If in doubt, treat events as dependent.

simple arithmetic. Reach for the coin again. Say you are going to toss it three times. What is the probability of only two heads? The set of all possible outcomes is as follows (where, for example, one outcome from three tosses of the coin is three heads, or HHH):

HHH THH HTH HHT TTH HTT THT TTT

Of the eight equally possible outcomes, only three involve two heads. There is a 3 in 8 chance of two heads. The probability is $3/8$, or 0.375. Look at this another way. Each outcome has a $1/8 = 0.125$ chance of happening, so the probability of two heads can also be found by addition: $0.125 + 0.125 + 0.125 = 0.375$.

The likelihood of not getting two heads can be computed in one of three ways. Either of the two approaches outlined may be used. But a simple method is to remember that since probabilities must sum to one, failure to achieve two heads must be $1 - 0.375 = 0.625$.

This has highlighted two important rules:

- If there are a outcomes where event A occurs and n outcomes in total, the probability of event A is calculated as $a \div n$. The shorthand way of writing this relationship is $P(A) = a \div n$.
- The probability of an event not occurring is equal to the probability of it happening subtracted from one. In shorthand: $P(\text{not } A) = 1 - P(A)$.

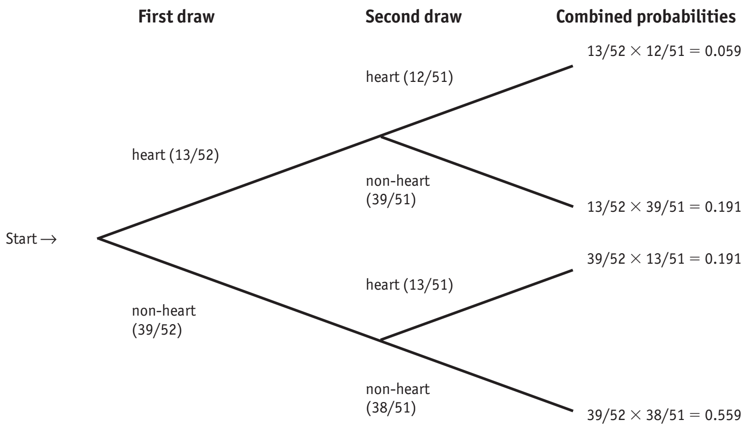
Assigning probabilities by observation. When probabilities cannot be estimated using the foresight inherent in the coin-tossing approach, experience and experiment help. If you know already that in every batch of 100 widgets 4 will be faulty, the probability of selecting a wobbly widget at random is $4 \div 100 = 0.04$. If 12 out of every 75 shoppers select Fat Cat Treats, there is a $12 \div 75 = 0.16$ probability that a randomly selected consumer buys them.

Subjective probabilities. On many occasions, especially with business problems, probabilities cannot be found from pure logic or observation. In these circumstances, they have to be allocated subjectively. You might, for example, say "considering the evidence, I think there is a 10% chance (ie, a 0.10 probability) that our competitors will imitate our new product within one year". Such judgments are acceptable as the best you can do when hard facts are not available.

Multiple events

1.3

The probability of a complex sequence of events can usually be found by drawing a little tree diagram. For example, the following tree shows the probability of selecting hearts on two consecutive draws (ie, without replacement) from a pack of 52 cards. Starting at the left, there are two outcomes to the first draw, a heart or a non-heart. From either of these two outcomes, there are two further outcomes. The probability of each outcome is noted, and the final probabilities are found by the multiplication rule. Check the accuracy of your arithmetic by noting that the probabilities in the final column must add up to one, since one of these outcomes must happen.



Composite events

There are some simple rules for deriving the probabilities associated with two or more events. You might know the risks of individual machines failing and want to know the chances of machine A or B failing; or the risk of both A and B breaking down at the same time.

The basic rules are summarised in the Probability rules box above. The following examples of how the rules work are based on drawing cards from a pack of 52 since this is relatively easy to visualise.

Composite events: **A** *given* **B**

Sometimes you want to know probabilities when you have some advance information. For example, if warning lights 4 and 6 are flashing, what are the chances that the cause is a particular malfunction? The solution is found by logically narrowing down the possibilities. This is easy to see with the playing card example.

The question might be phrased as “What is the probability that the selected card is a king given that a heart was drawn?”. The shorthand notation for this is P(K|H), where the vertical bar is read "given". Since there are 13 hearts and only one king of hearts the answer must be P(K|H) = $\frac{1}{13}$ = 0.077. This is known as conditional probability, since the card being a king is conditional on having already drawn a heart.

Revising probabilities as more information becomes available is considered in Chapter 8.

***Composite events:* A *or* B**

Add probabilities to find the chance of one event or another (eg, what are the chances of order A or order B arriving at a given time?).

Mutually exclusive events. The probability of drawing a heart or a black king is $\frac{13}{52} + \frac{2}{52} = 0.288$. Note that the two outcomes are mutually exclusive because you cannot draw a card which is both a heart and a black king. If a machine can process only one order at a time, the chances that it is dealing with order A or order B are mutually exclusive.

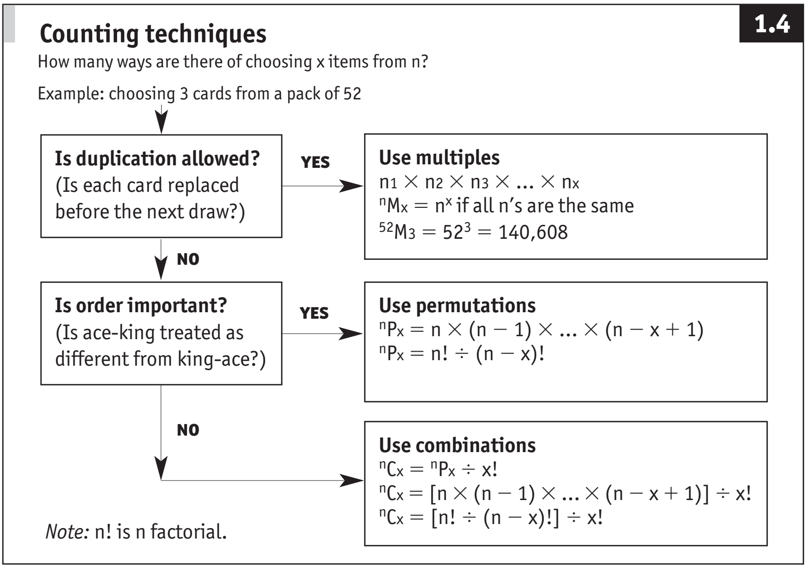
Non-exclusive events. The probability of drawing a heart or a red king is not $\frac{13}{52} + \frac{2}{52} = 0.288$. This would include double counting. Both the "set of hearts" and the "set of red kings" include the king of hearts. It is necessary to allow for the event which is double counted (the probability of drawing a heart and a red king). This is often done most easily by calculating the probability of the overlapping event and subtracting it from the combined probabilities. There is only one card which is both a king and a heart, so the probability of this overlapping event is $\frac{1}{52}$. Thus, the probability of drawing a heart or a red king is $\frac{13}{52} + \frac{2}{52} - \frac{1}{52} = 0.269$. If a machine can process several orders at a time, the chances that it is dealing with order A or order B are not mutually exclusive.

***Composite events:* A *and* B**

Multiply probabilities together to find the chance of two events occurring simultaneously (eg, receiving a large order and having a vital machine break down on the same day).

Independent events. The probability of drawing a red king from one pack and a heart from another pack is $\frac{2}{52} \times \frac{13}{52} = 0.01$. These two events are independent so their individual probabilities can be multiplied together to find the combined probability. A machine breakdown and receipt of a large order on the same day would normally be independent events.

Dependent events. Drawing from a single pack, a card which is both a red king and a heart is a dependent composite event. The probability is



easy to calculate if you think of this in two stages:

- What is the probability of drawing a red king ($2/52$)?
- Given that the card was a red king, what is the (conditional) probability that it is also a heart ($1/2$)?

These two probabilities multiplied together give the probability of a red king and a heart, $2/52 \times 1/2 = 1/52$. This answer can be verified by considering that there is only one card which meets both conditions so the probability must be $1/52$. A machine breakdown might be dependent on the receipt of a large order if the order overloads the machine. It is often difficult to decide whether events are independent or dependent. If in doubt, treat them as dependent.

Counting techniques

A frequent problem with probability is working out how many events are actually taking place. Visualise a card game. What are the chances that a five-card hand will contain four aces? It is the number of five-card hands containing four aces, divided by the total number of possible five-card hands. The two numbers that go into solving this equation are slightly elusive. So are the numbers required to solve some business

Ten factorials

0! =	1 =	1
1! =	1 =	1
2! =	2 × 1 =	2
3! =	3 × 2 × 1 =	6
4! =	4 × 3 × 2 × 1 =	24
5! =	5 × 4 × 3 × 2 × 1 =	120
6! =	6 × 5 × 4 × 3 × 2 × 1 =	720
7! =	7 × 6 × 5 × 4 × 3 × 2 × 1 =	5,040
8! =	8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 =	40,320
9! =	9 × 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 =	362,880

problems, such as "how many different ways are there that I can fulfil this order?".

There are three counting techniques (multiples, permutations and combinations – see Figure 1.4) which solve nearly all such problems.

Multiples

The multiples principle applies where duplication is permitted. A health clinic gives its patients identification codes (IDs) comprising two letters followed by three digits. How many IDs are possible? There are 26 characters that could go into the first position, 26 for the second, 10 for the third, 10 for the fourth, and 10 for the fifth. So there must be 26 × 26 × 10 × 10 × 10 = 676,000 possible IDs.

Think of each position as a decision. There are 26 ways to make the first decision, 26 ways to make the second decision, and so on through to ten ways to make the fifth decision. This may be generalised as n_1 ways to make decision D_1, n_2 ways to make D_2, n_3 ways to make D_3 and so on. Thus, there are $n_1 \times n_2 \times n_3 \times ... \times n_x$ ways to make x decisions.

Powers. There is a handy shortcut for situations where there are the same number of options for each decision. How many sequences are generated by nine tosses of a coin? (HTHHTTTHH is one sequence.) There are two ways for each decision: $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^9 = 512$. In this case, the multiples principle may be written ${}^nM_x = n^x$ for short, where ${}^nM_x$ is read as the number of multiples of x items selected from a total of n.

Combinations and permutations **1.5**

Arrangements of the 1 (ace), 2, 3 and 4 of hearts

Combinations	*Permutations*					
1,2,3	1,2,3	1,3,2	2,1,3	2,3,1	3,1,2	3,2,1
1,2,4	1,2,4	1,4,2	2,1,4	2,4,1	4,1,2	4,2,1
1,3,4	1,3,4	1,4,3	3,1,4	3,4,1	4,1,3	4,3,1
2,3,4	2,3,4	2,4,3	3,2,4	3,4,2	4,2,3	4,3,2

↑ 3 items in each row

↑ $3! = 3 \times 2 \times 1 = 6$ permutations for each set of 3

Permutations

With multiples the same values may be repeated in more than one position. Permutations are invoked when duplication is not allowed. For example, four executives are going to stand in a row to be photographed for their company's annual report. Any one of the four could go in the first position, but one of the remaining three has to stand in the second position, one of the remaining two in the third position, and the remaining body must go in the fourth position. There are $4 \times 3 \times 2 \times 1 = 24$ permutations or 24 different ways of arranging the executives for the snapshot.

Factorials. As previewed above, declining sequences of multiplication crop up often enough to be given a special name: factorials. This one ($4 \times 3 \times 2 \times 1$) is 4 factorial, written as 4! Good calculators have a key for calculating factorials in one move. The first ten are shown in the box. Incidentally, mathematical logic seems to be wobbly with 0! which is defined as equal to 1.

Permutations often involve just a little more than pure factorials. Consider how many ways there are of taking three cards from a pack of 52. There are 52 cards that could be selected on the first draw, 51 on the second and 50 on the third. This is a small chunk of 52 factorial. It is 52! cut off after three terms. Again, this can be generalised. The number of permutations of n items taken x at a time is ${}^{n}P_{x} = n \times (n - 1) \dots (n - x + 1)$. The final term is $(52 - 3 + 1 = 50)$ in the three-card example. This reduces to $52! \div 49!$ or ${}^{n}P_{x} = n! \div (n - x)!$ in general; which is usually the most convenient way to deal with permutations:

$${}^{52}P_{3} = \frac{52 \times 51 \times 50 \times 49 \times 48 \times \dots \times 2 \times 1}{49 \times 48 \times \dots \times 2 \times 1}$$

Combinations

Combinations are used where duplication is not permitted and order is not important. This occurs, for example, with card games; dealing an ace and then a king is treated as the same as dealing a king followed by an ace. Combinations help in many business problems, such as where you want to know the number of ways that you can pick, say, ten items from a production batch of 100 (where selecting a green Fat Cat Treat followed by a red one is the same as picking a red followed by a green).

To see how the combinations principle works, consider four playing cards, say the ace (or 1), 2, 3 and 4 of hearts:

- The four combinations of three cards that can be drawn from this set are shown on the left-hand side of Figure 1.5.
- The six possible permutations of each combination are shown on the right-hand side of Figure 1.5.
- For each set of 3 cards, there are $3! = 3 \times 2 \times 1 = 6$ permutations, or in general for each set of x items there are x! permutations.
- So the total number of permutations must be the number of combinations multiplied by x!, or $4 \times 3! = 24$. In general-purpose shorthand, ${}^nP_x = {}^nC_x \times x!$
- This rearranges to ${}^nC_x = {}^nP_x \div x!$; the relationship used to find the number of combinations of x items selected from n. In other words, when selecting 3 cards from a set of 4, there are $24 \div 3! = 4$ possible combinations.

The card puzzle

The puzzle which introduced this section on counting techniques asked what the chances were of dealing a five-card hand which contains four aces. You now have the information required to find the answer.

1. How many combinations of five cards can be drawn from a pack of 52? In this case, $x = 5$ and $n = 52$. So ${}^{52}C_5 = [52! \div (52 - 5)!] \div 5! = 2{,}598{,}960$. There are 2.6m different five-card hands that can be dealt from a pack of 52 cards.
2. How many five-card hands can contain four aces? These hands must contain four aces and one other card. There are 48 non-aces, so there must be 48 ways of dealing the four-ace hand:

$${}^4C_4 \times {}^{48}C_1 = [(4! \div 0!) \div 4!] \times [(48! \div 47\,! \div 1!] = 48$$

Modern counting systems

These systems work well because unlike, say, Roman numerals, they are positional and include the concept of zero. Figure 1.1 shows how the position or place of each digit (tens, hundreds, thousands, etc) contributes to the overall value of the number. Throw in the concept of negative numbers, and the decimal point to cater for fractional values, and you have a highly versatile system.

3 The probability of dealing a five-card hand containing exactly four aces is $48/2{,}598{,}960 = 0.0000185$ or 1 in 54,145 – not something to put money on.

The numerical base

Decimal

So far, this chapter has focused on the familiar decimal or base 10 system, which must surely have developed from our tendency to count on our fingers. A base, or radix, is essentially the number of digits or symbols used in a counting system. In base 10 there are ten digits, 0–9. We cycle through these, building overall values from right to left. Starting from nothing, the first item is counted as 1. The next is 2. When we pass the ninth item, we run out of digits, so we revert to zero in the rightmost position, and add 1 (ie, one lot of ten) to the left of the zero, and so on. Think about an electronic display counting quickly. The digits on the right will flash past, the numbers towards the left will increase ever more slowly, ticking off tens, hundreds, thousands, etc. Note how each position is just a power of 10, increasing as we move left.

Table 1.5 illustrates the decimal number 123,206. The top row indicates the place of each digit. The second row shows each place's value as an exponential. The third row is simply the place values in non-scientific notation. The bottom row shows the digit in each place multiplied by the place value.

Duodecimal

Base 10 is not the only way. Other bases have varying usage and popularity, especially base 12 (duodecimal or dozenal). Examples are 12 inches to a foot and the former British currency, which had 12 pennies to a shilling. These are vestiges of the Romans' use of 12; fractions are

Table 1.5 **Place values in decimal numbers**

Place	5	4	3	2	1	0
Place value (Base^{place})	10^5	10^4	10^3	10^2	10^1	10^0
Place value	100,000	10,000	1,000	100	10	1
Digits	**1**	**2**	**3**	**2**	**0**	**6**
Digit × place value	100,000	20,000	3,000	200	0	6

Note: 100,000 + 20,000 + 3,000 + 200 + 0 + 6 = 123,206

simple, because halves, quarters and thirds of a dozen are whole numbers. There is also a certain natural logic reflecting the 12 months in the lunar year, as well as the finger-counting system (still found in parts of Asia), which uses the thumb to tick off the 12 finger bones on that hand. By using the other hand to count off the dozens in the same way, you can count to 144.

For any base, the positional values are increasing powers of that base moving from right to left. In base 12, the number 10 means one lot of 12 and no units; this value is known as a dozen. Similarly, in base 12, 100 represents 12 dozens (a gross). The next landmark, 1,000, is 12 gross (a long gross). Most counting systems do not have names such as these for powers of the radix. Dozen, gross and long gross highlight the historical significance of base 12 in trade and commerce.

One other important point: base 12 requires a dozen single-digit symbols (since each symbol is positional) representing zero to 11. Mathematicians solve this by using various characters, such as A for 10 and B for 11. For example, the base 12 value 1A represents one dozen plus ten units; ie, 22 decimal. Incidentally, digit, although derived from the Latin word for our (ten) fingers and thumbs, is now applied to all the symbols used in any positional counting system.

Other bases

Another significant counting system is base 20 (vigesimal), used by the Mayans, who presumably counted on their fingers and toes (traces abound: 20 shillings to a pound, a score). The Babylonians favoured base 60 (sexagesimal) and had 60 different symbols representing the digits. We do not use their symbols, but base 60 is still used for counting minutes and seconds. More recently, base 2 (binary), base 8 (octal) and base 16 (hexadecimal, hex) have become widely used in connection with computers. As with duodecimal, hex requires more than ten symbols, so we use 0–9 followed by A–F. A represents decimal 10, while F stands in for decimal 15. Hex 1F is, therefore, 31 (one lot of 16 plus F (15).

Table 1.6 **A dozen-and-a-half, but who is counting**

Decimal	*Binary*	*Octal*	*Duodecimal*	*Hex*
1	000001	1	1	1
2	000010	2	2	2
3	000011	3	3	3
4	000100	4	4	4
5	000101	5	5	5
6	000110	6	6	6
7	000111	7	7	7
8	001000	10	8	8
9	001001	11	9	9
10	001010	12	A	A
11	001011	13	B	B
12	001100	14	10	C
13	001101	15	11	D
14	001110	16	12	E
15	001111	17	13	F
16	010000	20	14	10
17	010001	21	15	11
18	010010	22	16	12

Binary

Binary is a simple and robust counting system used by digital computers. It has just two digits, 0 and 1. These can be portrayed by electrical current flowing along a wire or not flowing. On or off. No room for argument. The position of each **bi**nary digi**t** (bit) is just a power of 2, so the arithmetic is simple when working with binary numbers (see Table 1.6).

Table 1.7 shows eight digits/bits. There is no formal definition of a byte (which is a group of bits), but 8-bit microprocessors used in the microcomputers of the 1970s popularised the idea of 8 bits to a byte. You can see that a byte of this size can be used to count up to 255 decimal (binary 11111111), which together with zero makes possible 256

Table 1.7 **Place values in binary numbers**

Place	7	6	5	4	3	2	1	0
Place value (baseplace)	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Place value	128	64	32	16	8	4	2	1
Digits	**1**	**1**	**0**	**0**	**1**	**1**	**1**	**0**
Digit × place value	128	64	0	0	8	4	2	0

Note: binary 11001110 = decimal 128 + 64 + 8 + 4 + 2 = 206

values. This was the foundation for the ASCII codes for data exchange, with which some readers may be familiar, in which various letters, numbers, symbols and control codes are represented by a decimal value between 0 and 255.

Octal and hex

One problem with binary is that writing even small numbers manually is error-prone. It happens that both base 8 and base 16 are powers of 2, and it is mind-bogglingly easy to convert between these two bases and binary. As a result, they are widely used in the computer industry.

Octal to binary. Since $8 = 2^3$, if you break a binary number into groups of three, and write down the octal equivalent for each group, you have the number in octal. Using the value 11001110 in Table 1.7:

Binary 011 001 110

$(0 + 2 + 1)\ (0 + 0 + 1)\ (2^2 + 2 + 0)$

Octal 3 1 6

Octal 316 = decimal: $(3 \times 64) + (1 \times 8) + 6 = 206$

Binary to hex. The arithmetic is exactly the same going from binary to hex, except that $16 = 2^4$, so you break the binary number into groups of four. The following example does this via decimal to make it easier to follow on first encounter:

Binary 1100 1110

Decimal $(2^3 + 2^2 + 0 + 0)$ $(2^3 + 2^2 + 2 + 0)$

$(8 + 4 = 12)$ $(8 + 4 + 2 = 14)$

Hex (decimal 12 = hex C) (decimal 14 = hex E)

C E

Hex CE = decimal: $(12 \times 16) + 14 = 206$

See Table 1.6 for a reminder about counting in hex.

Converting between bases

To convert from any base to decimal. Tables 1.6 and 1.7 show how to convert from any base into decimal: simply multiply each digit by its place value and add together the results.

To convert decimal to another base. The second easiest way to do this is to keep dividing by the target base, and write down the remainders in reverse order. For example, to convert decimal 206 into base 8:

206 / 8 = 25 r **6**

25 / 8 = 3 r **1**

3 / 8 = 0 r **3**

The remainders, in reverse order, are 316, which you will know from the above to be the octal equivalent of decimal 206. This is quite neat.

The easiest way to do conversions is to use a scientific calculator. Just click decimal, enter the decimal value, and click octal, or whatever.

IP addresses

Every device connected directly to the internet must have a unique identifier, known as an IP address. IP is shorthand for internet protocol, the rules or method (protocol) used for sending data from one device to another across the internet. There are many types of internet devices, including computers, tablets, cell phones and cameras.

An IP address is simply a numerical identifier. In 2013, most internet devices were using version 4 of the protocol, under which the IP address is a number between 0 and about 4.3 billion. At the time of writing, *The Economist*'s web server, economist.com, had the IP address 206.79.194.73. This is called dotted decimal notation, with four groups of decimal values separated by dots. The numbers can change, but while the human-friendly dotcom domain name remains the same, we do not have to learn and type awkward numbers. All IPv4 addresses are written in the same way.

In this IP addressing scheme, each group can have a decimal value between 0 and 255. As noted above (see Binary), decimal 256 corresponds to 2^8. In other words, an IP address comprises 4 bytes of 8 bits each, or 2^{32}. If you convert each value to binary and write the four results without a gap, you have the binary equivalent of the IP address:

206.79.194.73

11001110 01001111 11000010 01001001

Or, going in the other direction, if you slice the binary representation of an IP address into four parts, convert each one to decimal, and write them with dots, you have the dotted decimal version of an IP address. This makes it easy to convert to and from the values that computers use.

Clearly, 4.3 billion IP addresses is nowhere near enough for all the gizmos that we want to have on the internet, and the central pool of IPv4 addresses ran out on January 31st 2011. Fortunately, a new protocol, IPv6 provides over an octillion (10^{28}) of IP addresses for every person on the planet. IPv6 has a dotted hex notation, which comprises eight groups of hex values similar to the following:

2001:0db8:85a3:0042:1000:8a2e:0370:7334

This looks complicated, but it is easy to convert from hex to binary. It is the same as dotted decimal: convert each group from hex to binary and concatenate the results.

IPv6 has a 128-bit space for addresses (2^{128}, which is over 340 undecillion – an undecillion is 10^{38} – in decimal, although the quantity of usable addresses is a bit less). The same amount of space is reserved for routing, which will allow smart things to happen, such as mobile devices jumping between networks, but that is another story. And in case you were wondering, IPv5 was used for the internet stream protocol, but it did not work out.

Encryption

Encryption relies on mathematical algorithms to scramble text so that it appears to be gibberish to anyone without the key (essentially, a password) to unlock or decrypt it. You do not need to understand encryption to use it, but it is an interesting topic which deserves a mention here. There are two techniques:

- **Symmetric encryption** uses the same key to encrypt and decrypt messages.
- **Asymmetric encryption** involves the use of multiple keys (it is also known as dual-key or public-key encryption). Each party to an encrypted message has two keys, one held privately and the other known publicly. If I send you a message, it is encrypted using my private key and your public key. You decrypt it using your private key.

For a given amount of processing power, symmetric encryption is stronger than the asymmetric method, but if the keys have to be transmitted the risk of compromise is lower with asymmetric encryption. It is harder to guess or hack keys when they are longer. A 32-bit key has 2^{32} or over 4 billion combinations. Depending on the encryption technique and the application, 128-bit and 1024-bit keys are generally used for internet security.

To provide a flavour of the mathematics involved, consider the most famous asymmetric encryption algorithm, RSA (named for its inventors Ron Rivest, Adi Shamir, and Leonard Adleman). It works as follows:

1. Take two large prime numbers, p and q, and compute their product $n = pq$ (a prime number is a positive integer greater than 1 that can be divided evenly only by 1 and itself).
2. Choose a number, e, less than n and relatively prime to $(p-1)(q-1)$, which means e and $(p-1)(q-1)$ have no common factors except 1.
3. Find another number d such that $(ed-1)$ is divisible by $(p-1)(q-1)$.

The values e and d are called the public and private exponents, respectively. The public key is the pair (n, e); the private key is (n, d). Key length, as discussed above, relates to n, which is used as a modulus (a divisor where the result of integer division is discarded and the remainder is kept – see MOD in the A–Z).

The characters of the alphabet are assigned numerical values. I can send you a ciphertext c of any character value m, using your public key (n, e); where $c = m^e \bmod n$. To decrypt, you exponentiate $m = c^d \bmod n$. The relationship between e and d ensures that you correctly recover m. Since only you know d, only you can decrypt this message. Hackers note: it is difficult to obtain the private key d from the public key (n, e), but if you could factor n into p and q, then you could obtain the private key d.

Practical considerations

Choosing. Choosing an encryption algorithm requires technical knowledge. For asymmetric encryption, RSA remains the standard. For symmetric encryption, two acronyms to be familiar with are DES and AES. The original American government data encryption standard (DES) was developed back in 1972. Rocketing computer power has rendered it vulnerable to brute-force attacks (which step through a huge series of passwords until one unlocks the system). If you are still using DES in your organisation, it needs replacing. The American government standard since 2001 is the creatively named advanced encryption algorithm (AES). Experts generally think that AES with 256-bit keys is secure for the foreseeable future, although it is believed that the US National Security Agency may be able to access 128-bit AES, as well as RSA and the derivative used in encrypted websites (transport layer security/secure sockets layer – TLS/SSL).

Using. The latest versions of mainstream encryption-software products using RSA or AES are probably as secure as you can hope for. The trick is to use the longest and most complex keys/passwords you can manage. Do not write them on a sticky note "hidden" within reach of the computer. Then guard against tricks that compromise your password, such as keystroke loggers, side-channel attacks, and so on. Avoid obvious or guessable passwords, including the world's most common (which include password, 12345678, qwerty, abc123, monkey, iloveyou, sports terms, common names, etc). A mix of upper- and lower-case letters, numbers and symbols will significantly reduce the likelihood of a successful brute-force attack.